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Mohammadnejad, Mehrdad, Lednicka, Barbara, Gheibi, Mohammad and Behzadian Moghadam, Kourosh ORCID logo <https://orcid.org/0000-0002-1459-8408> (2026) Size-dependent thermal buckling analysis of functionally graded nanobeams using a Fredholm integral solution based on Eringen's nonlocal elasticity theory. *Applied Sciences*, 16 (19).

<http://dx.doi.org/10.3390/app16199722>

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Article

Size-Dependent Thermal Buckling Analysis of Functionally Graded Nanobeams Using a Fredholm Integral Solution Based on Eringen's Nonlocal Elasticity Theory

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Abstract

This study analyses the free vibration and thermal buckling of size-dependent, axially functionally graded Euler–Bernoulli nanobeams using a nonlocal thermo-elastic Fredholm integral model based on Eringen's nonlocal elasticity. The elastic modulus and mass density vary exponentially along the beam, so the thermal load, modeled as an equivalent axial force, follows the same exponential law. Four successive Fredholm integrations convert the governing equation into a weak-form integral equation that builds in the boundary conditions directly; this equation is then solved with a power series Galerkin method. The critical thermal buckling force is the axial force at which the fundamental frequency vanishes. Simple–simple (S–S), clamped–clamped (C–C), clamped–simple (C–S) and clamped–free beams are examined. Predicted frequencies agree with published solutions to within 0.11%. For a homogeneous beam, increasing the nonlocal parameter from zero to 0.2 lowers the fundamental frequency by 15.3% (S–S) and 18.3% (C–C). Shifting the thermal force from tension (–3) to compression (+3) lowers it by 26.9% (S–S) and 7.1% (C–C). Raising the nonlocal parameter from zero to 0.3 reduces the C–C buckling force by about 78% for a homogeneous beam and 94% for a gradient index of –2. Changing the gradient index from –2 to two lowers the first C–S frequency by 24.7% but the third by only 3.6%.

Keywords: size dependent; thermal buckling; nanobeams; Fredholm integral; thermal force

Received: 28 August 2026

Revised: 26 September 2026

Accepted: 27 September 2026

Published: 30 September 2026

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1. Introduction

FGMs are used in various categories of engineering and medical science, such as aerospace, medical, atomic reactions, and nanomaterial technology. The vibration and buckling of beams at the micro/nanoscale are affected significantly by small-size and size-dependent effects. Eringen was the first researcher to introduce his new beam theory that considered small-size and size-dependent effects, referred to as the Eringen's

nonlocal beam theory. This theory generalizes other specifications of classical theories and also introduces a new parameter in the governing differential equation for size-dependent effects as $(e_0 a)$. After Eringen's theory, other completed theories have been proposed by researchers, such as the couple stress theory, modified couple stress theory, strain gradient elasticity, doublet mechanics, and nonlocal strain gradient elasticity theory. Natural frequencies of functionally graded nanobeams with multiple cracks in a thermal environment have been determined [1]. The static stability of an elastic, incompressible nanorod subjected to an extensional force has been analyzed [2]. The dynamic vibration behavior of multilayer carbon nanotubes with stepped cross-sectional geometries under various boundary conditions has been comprehensively investigated [3]. An exact analytical investigation into the static response of helical single-walled carbon nanotube (SWCNT) beams based on Eringen's differential nonlocal elasticity theory has been presented [4]. This paper captured nanoscale effects arising from interatomic interactions. A new nonlocal nonlinear Euler–Bernoulli theory has been proposed to model the mechanical properties of nanobeams [5]. This paper considered the small-scale effect, thermal stress, and the temperature dependence of Young's modulus. An analytical method has been presented for the linear vibrations and buckling of nanoplates in a hygro-thermal environment [6]. An improved axial pipe–soil interaction model alongside a sand softening model has been presented to simulate the variation in soil resistance during thermal buckling of partially embedded pipelines [7]. The advantages and disadvantages of the Thermal Fluid–Solid Interaction method in comparison with the Thermal Structural and Conjugated Heat Transfer methods have been compared and demonstrated [8]. The comprehensive molecular dynamics benchmark solutions have been proposed to verify the nonclassical continuum models for miniaturized beams under tension, bending, buckling, and free transverse vibration [9]. The size-dependent buckling and vibration behaviors of Functionally Graded Magneto-Electro-Thermo-Elastic (FGMETE) Timoshenko nanobeams have been explored [10]. A Ritz-type solution based on Chebyshev polynomials has been introduced to investigate the effect of hygro-thermal conditions on the buckling and free vibration behaviors of functionally graded beams [11]. A finite element framework for buckling analysis based on the modified couple stress theory (MCST), applicable to arbitrary geometries, has been developed [12]. A theoretical framework has been developed to investigate the buckling behavior of one- and two-dimensional flexible structures, namely elastic rods and crystalline membranes, particularly when they experience large thermal fluctuations [13]. The thermal buckling of organic nanobeams resting on a viscous elastic foundation has been investigated [14]. A constraint-reformulated framework optimizing global compliance for multi-material structures under simultaneous thermo-mechanical loads and buckling constraints has been introduced [15]. The determination of buckling loads and mode shapes for periodic reticulated linear elastic structures has been formulated [16]. An interval spline finite point method has been presented to investigate the mechanical behavior of a functionally graded material nanobeam with defects [17]. The modified boundary conditions for the well-established differential model of Eringen's nonlocal elasticity theory have been proposed to deal with the free edges of the structural components [18]. Analytical solutions for the vibration of bidirectional functionally graded (BDFG) nanobeams have been proposed, and the nonlocal elasticity theory of Eringen has been followed to model the small-scale effects [19]. A stress-driven model incorporating surface energy effects has been introduced to investigate the bending response of Bernoulli–Euler nanobeams made from a functionally graded (FG) material with different cross-sectional shapes [20]. Under the influence of axial forces and uniform temperature variations, the thermal buckling and post-buckling of composite beams reinforced with functionally graded multilayer graphene platelets (GPLs) resting

on nonlinear elastic foundations have been examined [21]. The natural frequencies and instability of laminated nanocomposite beams in thermally pre-/post-buckled configurations have been analyzed [22]. The critical buckling temperature (ΔT_{cr}) of an offshore pipeline model from one-dimensional and three-dimensional finite element analyses has been obtained [23]. Using FE formulation, the combined effect of porosity and temperature on the buckling and vibration attributes of FG sandwich beams in the thermal environment has been investigated [24]. A comprehensive study has been established to investigate the thermal buckling instability and the subsequent post-critical deflection of a rotating nanocomposite microbeam reinforced with graphene platelets [25]. The instability and post-buckling behavior of a rotating nanocomposite beam subjected to a uniform thermal environment were investigated using the Chebyshev–Ritz method [26]. The thermal buckling and vibration of FG nanobeams have, in fact, received considerable attention. Ebrahimi and Salari [27] analyzed nonlocal FG nanobeams with temperature-dependent properties under uniform, linear and nonlinear temperature rises using Navier and differential transform solutions. Lal and Dangi [28] studied the thermo-mechanical vibration of bidirectional FG non-uniform Timoshenko nanobeams using the generalized differential quadrature method. Fang et al. [29] examined rotating nonlocal FG nanobeams in a thermal environment with a Galerkin scheme. Xu et al. [30] obtained buckling loads of FG nanobeams under a non-uniform temperature using stress-driven nonlocal elasticity. Nonlocal strain gradient models have been extended to magneto-thermal [31] and magneto-electro-elastic porous [32] FG nanobeams. However, these studies share three features. First, the material is graded through thickness, and the thermal load is uniform along the beam. Second, the governing equations are solved in a strong form with Navier, series, quadrature, or finite element techniques. Third, vibration and buckling are usually treated as separate problems. Axially graded nanobeams have been studied mainly without thermal loading [33], and integral equation formulations have so far been applied only to homogeneous nanobeams without thermal effects [34]. The specific gap addressed here is therefore the thermal buckling and vibration of axially graded nanobeams in which the equivalent thermal force itself varies along the beam. The proposed Fredholm integral approach differs from the studies above in three ways. It converts the variable-coefficient governing equation into a weak form in which the boundary conditions are built in through physically meaningful integration constants. It provides closed-form kernels for four support conditions. And a single eigenvalue formulation yields both the natural frequencies and the critical thermal buckling force.

Most thermal studies of FG nanobeams assume thickness-wise gradation and uniform thermal loads, and treat vibration and buckling separately. Here, the temperature change is modeled as an equivalent axial force that, owing to the exponential axial variation in the elastic modulus, varies exponentially along the beam. This force enters the Euler–Bernoulli equation with Eringen’s nonlocal elasticity, forming the nonlocal thermo-elastic model. Four successive Fredholm integrations then convert this equation into a weak form whose integration constants embed the boundary conditions. The novelty lies in this unified framework: closed-form weak forms for four support types, with one eigenvalue formulation giving both natural frequencies and the critical thermal buckling force.

Despite the extensive literature on the thermal buckling and free vibration of functionally graded nanobeams, most existing studies employ direct numerical or semi-analytical techniques, such as finite element, differential quadrature, Ritz, Chebyshev–Ritz, spectral, and direct Galerkin methods, to solve the fourth-order governing differential equations. Semi-analytical solution frameworks based on Fredholm integral transformations have received comparatively little attention for nonlocal functionally

graded nanobeams subjected to thermal loading. The novelty of the present study lies in developing a Fredholm integral solution framework for the governing equation derived from Eringen's differential nonlocal elasticity theory. By transforming the fourth-order differential equation into an equivalent weak-form integral equation through successive integrations, the proposed approach incorporates the boundary conditions naturally via integration constants and eliminates the need to approximate higher-order derivatives directly. Approximating the unknown function itself rather than its higher-order derivatives leads to improved convergence with relatively few expansion terms. This formulation provides a simple and efficient semi-analytical framework with good convergence characteristics while maintaining excellent agreement with benchmark solutions available in the literature.

Unlike strong-form Navier, GDQ or FE studies with uniform thermal loads, this work contributes: (i) an Eringen nonlocal Euler–Bernoulli model of axially graded nanobeams, $E, \rho \propto \exp(c\xi)$, under an axially varying thermal force $N(x) = -E_0 A \alpha_t \Delta \theta \exp(cx/L)$; (ii) four successive Fredholm integrations giving closed-form weak-form kernels $f_i(\xi, s)$ for S–S, C–C, C–S and C–F beams; and (iii) one Galerkin eigenproblem, $\det[B(\lambda, p)] = 0$, yielding λ for prescribed p and p_{cr} at $\lambda = 0$.

2. Methodology

The methodology of this research is based on a Fredholm integral transformation approach to determine the vibration characteristics of functionally graded material (FGM) nanobeams with size-dependent effects. The study begins by formulating the governing partial differential equation of motion using the Euler–Bernoulli beam theory augmented by Eringen's nonlocal elasticity theory to capture small-scale effects. By assuming harmonic motion, the governing equation is reduced to an ordinary differential equation in spatial coordinates. Instead of solving the resulting fourth-order differential equation directly (strong form), the equation is systematically transformed into a weak (integral) form through successive Fredholm integrations. Boundary conditions are incorporated naturally during this transformation process via integration constants. The weak-form integral equation is then approximated using a power series representation, which allows the problem to be converted into a system of linear algebraic equations through the Galerkin weighted residual method. The non-trivial solution of this system yields the eigenvalues, corresponding to the natural frequencies of the nanobeam. The methodology adopts the Fredholm integral equation approach, which is an efficient mathematical tool for solving linear integral equations. The general Fredholm integral equation consists of a kernel function, a known function, and an unknown target function. The main objective is to determine the unknown function that satisfies the integral equation. In this study, the governing differential equation of motion is first reformulated into this integral framework, enabling semi-analytical tractability and numerical efficiency.

The governing equation is derived based on the classical Euler–Bernoulli beam theory, assuming small deformations and neglecting shear deformation effects. To account for size-dependent behavior at the nanoscale, Eringen's nonlocal elasticity theory is incorporated into the formulation. This leads to a partial differential equation of motion describing the transverse vibration of the FGM nanobeam.

A harmonic form is assumed for the transverse displacement, which separates the temporal and spatial variables. This assumption converts the partial differential equation into an ordinary differential equation with respect to the spatial coordinate. The resulting equation involves a fourth-order derivative of the mode shape function and is referred to as the strong form of the governing equation.

To avoid the direct solution of the fourth-order differential equation, the strong form is transformed into a weak (integral) form through four successive Fredholm integrations. Each integration introduces an integration constant, which has a clear physical interpretation related to shear force, bending moment, slope, and deflection of the beam. This step significantly simplifies the mathematical treatment while maintaining physical consistency.

The integration constants appearing during the transformation process are determined using the boundary conditions applied at both ends of the nanobeam. Each integral equation obtained at a specific integration stage corresponds to a particular boundary condition, such as shear force, bending moment, slope, or transverse displacement. This systematic incorporation of boundary conditions ensures an accurate and consistent formulation of the weak-form equation.

After determining the integration constants, the mode shape function remains as the only unknown in the final weak-form integral equation. This function is approximated using a power series expansion, which transforms the integral equation into an algebraic form. This approximation facilitates further numerical treatment while preserving solution accuracy.

The Galerkin weighted residual method is employed to convert the power series representation into a system of linear algebraic equations. By enforcing the orthogonality of the residuals, the method leads to a characteristic eigenvalue problem. The non-trivial solutions of this system correspond to the eigenvalues, which represent the natural frequencies of the FGM nanobeam under nonlocal effects. The present formulation is based on the Euler–Bernoulli beam theory. This choice is motivated by the primary objective of the study, namely the development of a Fredholm integral solution framework for the governing equation derived from Eringen’s differential nonlocal elasticity theory. Since the nanobeams considered herein have relatively large length-to-thickness ratios, the influence of transverse shear deformation is assumed to be negligible, making the Euler–Bernoulli beam model an appropriate approximation. While higher-order beam theories, such as the nonlocal Timoshenko beam theory, may provide greater accuracy for moderately thick nanobeams by accounting for transverse shear deformation and rotary inertia, incorporating such effects is beyond the scope of the present study and represents a promising direction for future research.

3. Free Vibration Analysis of FGM Nanobeams Under Thermal Effects

3.1. Statement of Problem and Governing Equations

A beam under temperature variations undergoes the expansion or contraction of its volume. This volume variation results in an axial deformation in the beam’s length as well as an overall deformation in the beam’s structure. When these temperature variations reach their critical limit, denoting thermal buckling, instability occurs in the beam. Thermal buckling is an important phenomenon in nano-mechanics. Herein, it is assumed that the nanobeam is in a thermal environment, and an increase or decrease in the beam’s length is modeled by an equivalent axial force caused by thermal effects acting on the cross-section of the nanobeam. According to the thermodynamic principle, the thermal axial force N_t is obtained as follows:

$$N_t = -\frac{A\alpha\Delta T L}{\int_0^L \frac{dx}{E(x)}} \quad (1)$$

In Equation (1), A is the beam’s cross-sectional area, α is the thermal expansion coefficient of the beam’s material, ΔT is the temperature variation, and $E(x)$ is

the elasticity modulus of the beam's material. The details of the thermal axial force calculation are presented in Appendix A. The material properties, such as the elasticity modulus and volume density, are assumed to vary along the beam's length as follows:

$$\begin{cases} E(x) = E_0 \times e^{c \frac{x}{L}} \\ \rho(x) = \rho_0 \times e^{c \frac{x}{L}} \end{cases} \quad (2)$$

where E_0 and ρ_0 are the modulus of elasticity and volume density of beam materials corresponding to $x=0$, and c is the gradient index denoting variation profiles of material properties. Figure 1 presents the variation profiles of the material properties for $c = -2, -1, 0, 1, 2$.

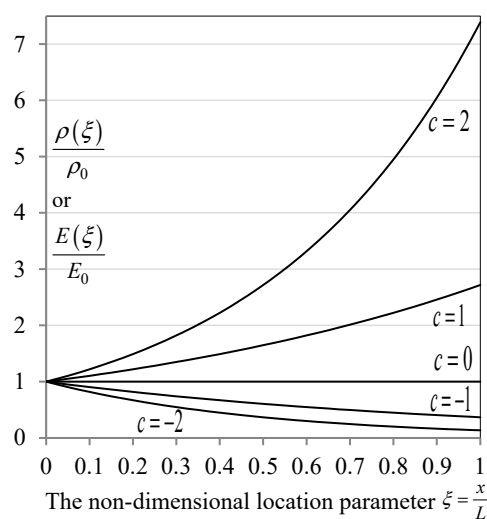


Figure 1. The variation profiles of the material properties for $c = -2, -1, 0, 1, 2$.

According to the theory of Zarzycki (1982) and Kittel (1983), the thermal expansion coefficient α_t can be calculated as [35]:

$$\alpha_t = \frac{\gamma_G \times C_V}{3} \times \frac{\rho(x)}{E(x)} = \frac{\gamma_G \times C_V \times \rho_0}{3E_0} \equiv \text{Constant} \quad (3)$$

in which γ_G is the Grüneisen constant and C_V is the specific heat capacity at a constant volume. According to the considered relations for $\rho(x)$ and $E(x)$ in Equation (2), the thermal expansion coefficient α_t is a constant parameter herein; see Equation (3). It should be noted that Equation (2) assumes that Young's modulus and mass density vary according to the same exponential gradation, while the coefficient of thermal expansion is taken to be constant throughout the beam. These assumptions are introduced to facilitate the development of a closed-form analytical solution based on the proposed Fredholm integral framework. Although such assumptions are widely adopted in analytical studies of functionally graded structures, they may not fully represent the behavior of general metal–ceramic FGMs, in which different material properties can exhibit distinct spatial gradation profiles and temperature-dependent characteristics. Therefore, the present formulation should be regarded as an analytical benchmark, whereas the incorporation of independent gradation functions and temperature-dependent material properties constitutes a worthwhile direction for future inves-

tigation. Introducing Equation (2) into Equation (1) results in the following function for thermal axial force:

$$N(x)=N_t=-\left(\frac{c}{1-e^{-c}}\right)E_0A\alpha\Delta T \quad (4)$$

To develop the problem, we consider an FGM nanobeam. The origin point of the coordinate axes is located at the left end of the beam. The x -axis is deemed to correspond to the longitudinal axis of the beam. The z -axis is considered to be the neutral axis of the beam section. Therefore, the lateral deflection of the beam is parallel to the y -axis. The modulus of elasticity and volume density on the left side of the beam are denoted as E_0 and ρ_0 . Figure 2 presents the described beam.

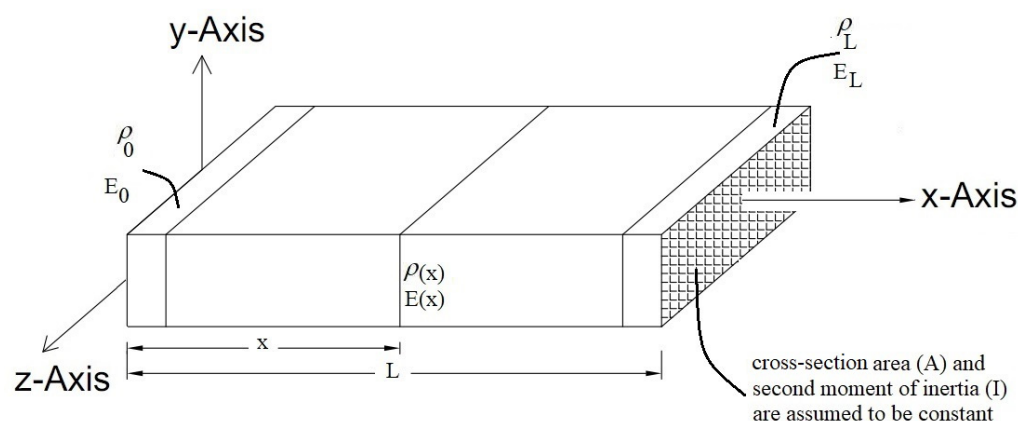


Figure 2. FGM nanobeam.

Using the Hamilton principle, the governing equation of motion for a Euler–Bernoulli beam is combined with Eringen’s nonlocal elasticity theory. The result is the following partial differential equation for the free vibration motion of an FGM nanobeam:

$$\frac{\partial^2}{\partial x^2} \left[E(x)I \frac{\partial^2 W(x,t)}{\partial x^2} \right] - (e_0a)^2 \frac{\partial^2}{\partial x^2} \left[\rho(x)A \frac{\partial^2 W(x,t)}{\partial t^2} \right] + (e_0a)^2 \frac{\partial^3}{\partial x^3} \left[N(x) \frac{\partial W(x,t)}{\partial x} \right] - \frac{\partial}{\partial x} \left[N(x) \frac{\partial W(x,t)}{\partial x} \right] + \rho(x)A \frac{\partial^2 W(x,t)}{\partial t^2} = 0 \quad (5)$$

in which $W(x,t)$ is the lateral deflection of the beam. I and A are the second moment of inertia and area of cross-section, which are constant parameters. $E(x)$ and $\rho(x)$ are functions of the elasticity modulus and volume density that vary along the x -axis. $N(x)$ is the thermal axial force. e_0a is a size-dependent parameter coming from Eringen’s nonlocal elasticity theory. To eliminate the time parameter from calculations, a harmonic vibration is considered for the lateral deflection of the beam:

$$W(x,t) = \bar{\varphi}(x) \times e^{j\omega t} \quad (6)$$

where $\bar{\varphi}(x)$ is the mode shape function and ω is the natural frequency of the vibration. Applying Equation (6) to Equation (5) transforms it into an ordinary differential equation based on the mode shape function and location parameter x :

$$\frac{d^2}{dx^2} \left[E(x) I \frac{d^2}{dx^2} \bar{\varphi}(x) \right] + (e_0 a)^2 \frac{d^2}{dx^2} \left[\rho(x) A \omega^2 \bar{\varphi}(x) \right] + (e_0 a)^2 \frac{d^3}{dx^3} \left[N(x) \frac{d}{dx} \bar{\varphi}(x) \right] - \frac{d}{dx} \left[N(x) \frac{d}{dx} \bar{\varphi}(x) \right] - \rho(x) A \omega^2 \bar{\varphi}(x) = 0 \quad (7)$$

We assume that the FGM nanobeam has a constant cross-section while the modulus of elasticity and volume density change along the beam's length. The modulus of elasticity, volume density, and thermal axial force follow an exponential function, as introduced in Equations (2) and (4). These equations are then introduced into Equation (7). The result is as follows:

$$E_0 I \frac{d^2}{dx^2} \left[e^{\frac{c}{L} x} \times \frac{d^2}{dx^2} \bar{\varphi}(x) \right] + (e_0 a)^2 \rho_0 A \omega^2 \frac{d^2}{dx^2} \left[e^{\frac{c}{L} x} \times \bar{\varphi}(x) \right] - (e_0 a)^2 N_t \frac{d^3}{dx^3} \left[\frac{d}{dx} \bar{\varphi}(x) \right] + N_t \frac{d}{dx} \left[\frac{d}{dx} \bar{\varphi}(x) \right] - \rho_0 A \omega^2 e^{\frac{c}{L} x} \times \bar{\varphi}(x) = 0 \quad (8)$$

For simpler calculations, the following non-dimensional parameters are considered:

$$\xi = \frac{x}{L}, \quad \varphi(\xi) = \frac{\bar{\varphi}(x)}{L}, \quad \psi = \frac{e_0 a}{L}, \quad p = \frac{N_t}{E_0 I}, \quad \lambda^2 = \frac{\rho_0 A L^4}{E_0 I} \times \omega^2 \quad (9)$$

In Equation (9), ξ is a non-dimensional parameter that converts the domain of the problem from $0 - L$ to $0 - 1$. This conversion makes the integration process simpler. $\varphi(\xi)$ is the non-dimensional mode shape function. ψ , p and λ are the non-dimensional parameters corresponding to the size-dependent effect, thermal axial force, and the natural frequency of the beam. Substitution of these dimensionless parameters into Equation (8) results in the following equation:

$$\frac{d^4 \varphi}{d\xi^4} + \alpha_1 \frac{d^3 \varphi}{d\xi^3} + \alpha_2 \frac{d^2 \varphi}{d\xi^2} + \alpha_3 \frac{d \varphi}{d\xi} + \alpha_4 \varphi(\xi) = 0 \quad (10)$$

where $\alpha_1, \alpha_2, \alpha_3$ and α_4 can be calculated as follows:

$$\alpha_1 = \frac{2c - 2c\psi^2 p}{1 - \psi^2 p}, \quad \alpha_2 = \frac{c^2 + \psi^2 \omega^2 - 3\psi^2 p c^2 + p}{1 - \psi^2 p}, \quad \alpha_3 = \frac{2\psi^2 \omega^2 c - \psi^2 p c^3 + p}{1 - \psi^2 p}, \quad \alpha_4 = \frac{\psi^2 \omega^2 c^2 - \omega^2}{1 - \psi^2 p} \quad (11)$$

We call Equation (10) the strong form of the governing differential equation and try to convert it into its weak form according to the Fredholm integral transformation.

3.2. Development of Weak-Form Integral Equations

According to the Fredholm integral transformation approach introduced in Section 2, we need successive integrations from the strong form of the differential equation until the derivatives of the function are eliminated in the last equation after the last integration process. Therefore, the last equation only contains the function itself. Accordingly, we need four repetitive integrations from both sides of Equation (10), with respect to ξ between 0 and ξ . These integrations are carried out according to the Fredholm transformation approach. The results of four successive integrations are the following differential equations:

$$\frac{d^3\varphi}{d\xi^3} + \alpha_1 \frac{d^2\varphi}{d\xi^2} + \alpha_2 \frac{d\varphi}{d\xi} + \alpha_3 \varphi(\xi) + \alpha_4 \int_0^\xi \varphi(s) ds = C_1 \quad (12)$$

$$\frac{d^2\varphi}{d\xi^2} + \alpha_1 \frac{d\varphi}{d\xi} + \alpha_2 \varphi(\xi) + \alpha_3 \int_0^\xi \varphi(s) ds + \alpha_4 \int_0^\xi (\xi-s) \varphi(s) ds = C_1 \xi + C_2 \quad (13)$$

$$\frac{d\varphi}{d\xi} + \alpha_1 \varphi(\xi) + \alpha_2 \int_0^\xi \varphi(s) ds + \alpha_3 \int_0^\xi (\xi-s) \varphi(s) ds + \frac{\alpha_4}{2} \int_0^\xi (\xi-s)^2 \varphi(s) ds = \frac{C_1}{2} \xi^2 + C_2 \xi + C_3 \quad (14)$$

$$\varphi(\xi) + \int_0^\xi f_1(\xi, s) \varphi(s) ds = \frac{C_1}{6} \xi^3 + \frac{C_2}{2} \xi^2 + C_3 \xi + C_4 \quad (15)$$

In Equation (15), the function $f_1(\xi, s)$ is calculated as follows:

$$f_1(\xi, s) = \alpha_1 + \alpha_2 (\xi-s) + \frac{\alpha_3}{2} (\xi-s)^2 + \frac{\alpha_4}{6} (\xi-s)^3 \quad (16)$$

Corresponding to each integration, one integration constant appears. These constants should be determined using appropriate boundary conditions at both ends of the beam. Equation (12) can be used for the shear force boundary condition. Equation (13) is used for the bending moment boundary condition, and Equations (14) and (15) are applied for slope and displacement boundary conditions, respectively. The beams with simple–simple (S–S), clamped–clamped (C–C), clamped–simple (C–S), and clamped–free (C–F) end conditions are investigated herein, and their corresponding relations are developed.

Boundary conditions: The mechanical boundary conditions have been obtained directly from Hamilton's principle. In the present work, the boundary conditions derived in Reference [1] have been employed directly. Since the detailed derivation and expressions of these boundary conditions have already been presented in the cited reference, they have not been repeated in the present manuscript for the sake of brevity. The variational formulation yields both the corresponding natural boundary conditions expressed through the generalized bending moment and generalized transverse shear force. For beams subjected to an axial force, the generalized shear force includes the additional contribution, leading to the free-end conditions $M = 0$ and $V = 0$. It should be emphasized that the present study adopts Eringen's differential nonlocal constitutive model. The boundary conditions corresponding to various beam end conditions are presented below [1]:

$$\left\{ \begin{array}{ll} \text{Simple – Simple :} & @ \xi = 0 \quad \varphi = 0, \quad @ \xi = 0 \quad \frac{d^2\varphi}{d\xi^2} = 0, \quad @ \xi = 1 \quad \varphi = 0, \quad @ \xi = 1 \quad \frac{d^2\varphi}{d\xi^2} = 0 \\ \text{Clamped – Clamped :} & @ \xi = 0 \quad \varphi = 0, \quad @ \xi = 0 \quad \frac{d\varphi}{d\xi} = 0, \quad @ \xi = 1 \quad \varphi = 0, \quad @ \xi = 1 \quad \frac{d\varphi}{d\xi} = 0 \\ \text{Clamped – Simple :} & @ \xi = 0 \quad \varphi = 0, \quad @ \xi = 0 \quad \frac{d\varphi}{d\xi} = 0, \quad @ \xi = 1 \quad \varphi = 0, \quad @ \xi = 1 \quad \frac{d^2\varphi}{d\xi^2} = 0 \\ \text{Clamped – Free :} & @ \xi = 0 \quad \varphi = 0, \quad @ \xi = 0 \quad \frac{d\varphi}{d\xi} = 0, \quad @ \xi = 1 \quad \frac{d^3\varphi}{d\xi^3} + \alpha_2 \frac{d\varphi}{d\xi} = 0, \quad @ \xi = 1 \quad \frac{d^2\varphi}{d\xi^2} = 0 \end{array} \right. \quad (17)$$

Boundary conditions (17) are applied to Equations (12)–(15), and the corresponding integration constants are determined for each case. Substituting these constants into Equation (15) yields the following integral equation:

$$\varphi(\xi) + \int_0^{\xi} f_1(\xi, s) \varphi(s) ds + \int_0^1 f_2(\xi, s) \varphi(s) ds = 0 \quad (18)$$

where

$$\left\{ \begin{array}{l} \text{for all boundary conditios: } f_1(\xi, s) = \alpha_1 + \alpha_2(\xi - s) + \frac{\alpha_3}{2}(\xi - s)^2 + \frac{\alpha_4}{6}(\xi - s)^3 \\ \text{Simple-Simpel: } f_2(\xi, s) = -g_3(s) \frac{\xi^3}{6} - \frac{\beta}{2} g_2(s) \xi^2 - \frac{\beta}{\alpha_1} g_2(s) \xi \\ \text{Clamped-Clamped: } f_2(\xi, s) = -g_1(s) \frac{\xi^3}{6} - g_2(s) \frac{\xi^2}{2} \\ \text{Clamped-Simple: } f_2(\xi, s) = -g_3(s) \frac{\xi^3}{6} - g_2(s) \frac{\xi^2}{2} \\ \text{Clmaped-Free: } f_2(\xi, s) = -g_3(s) \frac{\xi^3}{6} - g_4(s) \frac{\xi^2}{2} \end{array} \right. \quad (19)$$

The functions and parameters introduced in Equation (19) can be calculated using the relations provided in the Appendix A. We call Equation (18) the weak form of the governing equation for an FGM nanobeam. There are two unknown parameters in this equation. The first one is the non-dimensional parameter λ referring to the natural frequency of the beam, and the second one is the mode shape function $\varphi(\xi)$.

3.3. Solution: Weighted Residual Method

In Section 3.2, we converted the strong form of the differential equation of motion for free vibration of FGM nanobeams under thermal effects into its weak form. According to the Fredholm transformation approach, successive integrations were applied for the conversion of the strong form into the weak form. For various end conditions, weak-form integral equations were obtained with a general form as:

$$\varphi(\xi) + \int_0^{\xi} f_1(\xi, s) \varphi(s) ds + \int_0^1 f_2(\xi, s) \varphi(s) ds = 0 \quad (20)$$

It should be noted that in Equation (20), the function $f_1(\xi, s)$ is the same for all cases of boundary conditions, and the function $f_2(\xi, s)$ was determined specifically for each case. In Equation (20), there are two unknown parameters: the mode shape function $\varphi(\xi)$ and the non-dimensional eigenvalue parameter λ corresponding to the natural frequency of the vibration. The final aim of this paper is to determine the natural frequencies of the vibration. We use the weighted residual method to solve the integral Equation (20). The weighted residual method is a powerful classical method that is applied to a differential equation to obtain an approximate solution for it. In the integral Equation (20), $\varphi(\xi)$ is the unknown dependent function that we should seek an appropriate solution for it to satisfy Equation (20). An approximate solution is considered for $\varphi(\xi)$ and is introduced into Equation (20). Since the assumed solution is only an approximation, it does not, in general, satisfy Equation (20) and hence results in an error, or what we call a residual. The residual is then made to vanish over the entire so-

lution domain to produce a system of algebraic equations. We now seek the approximate solution for $\varphi(\xi)$ in the form of a generalized Fourier series as:

$$\varphi(\xi) = \sum_{i=0}^n c_i N_i(\xi) = c_0 N_0(\xi) + c_1 N_1(\xi) + c_2 N_2(\xi) + \dots + c_n N_n(\xi) \quad (21)$$

in which c_i are unknown coefficients called fitting coefficients, and n is the number of fitting coefficients. $N_i(\xi)$ are assumed to be linearly independent functions of ξ and are called trial functions. The trial functions can be polynomials, trigonometric functions, etc. One of the simplest choices for a trial function is a polynomial for a one-dimensional problem, which can be obtained by taking $N_i(\xi) = \xi^i$. Herein, we use a polynomial trial function to approximate the mode shape function. The result is:

$$\varphi(\xi) = \sum_{i=0}^n c_i \times \xi^i = c_0 + c_1 \xi + c_2 \xi^2 + \dots + c_n \xi^n \quad (22)$$

In this relation, n is a positive integer value that is selected by the user to control the accuracy of the results and the convergence rate of the solution. c_i ($i=0,1,\dots,n$) are fitting coefficients and ξ^i ($i=0,1,\dots,n$) are trial functions. The introduction of the approximate mode shape into the resulting integral Equation (20) transforms it into an integral series as follows:

$$R(\xi) = \sum_{i=0}^n \left[\xi^i + \int_0^\xi f_1(\xi, s) s^i ds + \int_0^1 f_2(\xi, s) s^i ds \right] c_i \quad (23)$$

where $R(\xi)$ is a measure of error commonly referred to as the residual. The residual is then made to vanish. Hence, the trial functions are multiplied by the integral series (23) and then integrated with respect to ξ from 0 to 1. $(0-1)$ is the non-dimensional domain of the problem corresponding to $(0-L)$ in the original domain. L is the beam length. Herein, ξ^i ($i=0,1,2,\dots,n$) are the trial functions. Accordingly, both sides of Equation (23) are multiplied by ξ^m ($m=0,1,2,\dots,n$) and then integrated with respect to ξ from 0 to 1. The result is as follows:

$$\int_0^1 \xi^m R(\xi) d\xi = \sum_{i=0}^n \left[\int_0^1 \xi^{m+i} d\xi + \int_0^1 \int_0^\xi f_1(\xi, s) s^i \xi^m ds d\xi + \int_0^1 \int_0^1 f_2(\xi, s) s^i \xi^m ds d\xi \right] c_i = 0 \quad (m=0,1,2,\dots,n) \quad (24)$$

According to the weighted residuals method, we force Equation (24) to vanish over the solution domain. Therefore, the right side of Equation (24) is equal to zero. Equation (24) can be rewritten as a system of linear algebraic equations:

$$\sum_{i=0}^n [G(m,i) + F_1(m,i) + F_2(m,i)] [c_i] = 0 \quad (m=0,1,2,\dots,n) \quad (25)$$

in which

$$\begin{cases} G(m,i) = \int_0^1 \xi^{m+i} d\xi \\ F_1(m,i) = \int_0^1 \int_0^1 f_1(\xi,s) s^i \xi^m ds d\xi \\ F_2(m,i) = \int_0^1 \int_0^1 f_2(\xi,s) s^i \xi^m ds d\xi \end{cases} \quad (26)$$

We present the system of Equation (25) in a compact matrix form as: $[B]_{(n+1,n+1)} \times [c_i]_{(n+1,1)} = 0$. After approximation of the mode shape function by the polynomial function (22), the only remaining unknown parameter in the constituent elements of $[B]_{(n+1,n+1)}$ is the non-dimensional eigenvalue parameter λ corresponding to the natural frequency of the vibration. The non-trivial solution is determined when the determinant of $[B]_{(n+1,n+1)}$ has to vanish—say $[B] = 0$. The non-trivial solution provides a polynomial function named as frequency equation. The roots of this equation are the eigenvalues of the vibration λ . In Equation (22), n is a given positive integer value named as the number of fitting coefficients. n , chosen by the user, controls the accuracy of the results and the convergence rate of the solution. The choice of a larger n enables the calculation of natural frequencies at higher modes, as well as increases the accuracy of the results obtained for lower modes. However, choosing a large n may require more time to obtain the results by a solver software, such as MATLAB (R2015b). This is due to the construction of a larger matrix for $[B]_{(n+1,n+1)}$.

4. Thermal Buckling Analysis

In Section 3, we developed the weak-form integral equations for free vibration analysis of FGM nanobeams with various end boundary conditions. The presented relations can be used to obtain the eigenvalues corresponding to the thermal buckling forces of FGM nanobeams. We considered the effects of thermal variations by an axial force acting on the cross-section of the beam as:

$$N(x) = N_t = -\left(\frac{c}{1-e^{-c}}\right) E_0 A \alpha \Delta T \quad (27)$$

The non-dimensional parameter $p = \frac{N_t}{E_0 I}$ referring to thermal axial force, was introduced to the governing equation of motion. When we want to obtain the natural frequencies, the parameter p is an input parameter. Therefore, $[B] = 0$ yields the non-dimensional eigenvalue λ . However, when the aim is the calculation of the thermal buckling force, we can adopt $\omega = 0 \rightarrow \lambda = 0$ and introduce $\lambda = 0$ into the obtained relations and consider the parameter p as an unknown parameter in the relations. Accordingly, in this case, $[B] = 0$ results in the non-dimensional parameter p . It

should be noted that in this case, the smallest root of the equation obtained from $\|B\|=0$ is the desired answer.

5. Results of Analysis

In this section, we first verify the proposed relations for the determination of the natural frequencies of the vibration of FGM nanobeams under thermal effects. Then, the results obtained from the thermal buckling analysis are presented and discussed.

5.1. Verification of the Results

To verify the proposed relations in this paper, we consider the assumptions considered by Refs. [1,36] in order to make it possible to compare the results. The effects of thermal force (P) and the nonlocal parameter (ψ) on the first natural frequency for the beam with simple–simple and clamped–clamped end conditions are presented in Table 1. The effect of non-homogeneity is not considered; therefore, the beam is considered uniform and homogeneous in this section of calculations, and $c=0$ is applied in the calculation of the first natural frequency. The results are compared with those obtained by Refs. [1,36]. It should be noted that the square root of the first natural frequency has been presented in the results of Ref. [1]. The results of Table 1 demonstrate that the nonlocal parameter (ψ) decreases the first natural frequency. On the other hand, the size-dependent effect decreases the first natural frequency. Also, when the temperature decreases, the first natural frequency increases.

Table 1. The first non-dimensional natural frequency for a uniform and homogeneous beam under the effects of thermal force and size-dependent effects ($c = 0$).

		$\psi=0$			$\psi=0.1$			$\psi=0.2$			
		p	Present	[1]	[36]	Present	[1]	[36]	Present	[1]	[36]
Simple-Simple	3	8.23695	8.23632	8.23460	7.68732	7.68675	7.68287	6.34583	6.34636	6.34133	
	2	8.81421	8.81377	8.80427	8.30295	8.30304	8.30016	7.07924	7.07879	7.07613	
	1	9.35652	9.35686	9.34830	8.87659	8.87622	8.87503	7.74419	7.74453	7.74230	
	0	9.8696	9.86965	9.86965	9.41588	9.41507	9.41446	8.35692	8.35672	8.35556	
	-1	10.3578	10.3580	10.3510	9.92637	9.92565	9.92502	8.92812	8.92814	8.92814	
	-2	10.8244	10.8247	10.8214	10.4123	10.4122	10.4109	9.4652	9.4648	9.4636	
	-3	11.2721	11.2721	11.2694	10.8768	10.8754	10.8735	9.9737	9.9735	9.9698	
		$\psi=0$			$\psi=0.1$			$\psi=0.2$			
		p	Present	[1]	[36]	Present	[1]	[36]	Present	[1]	[36]
Clamped-Clamped	3	21.5318	21.5333	21.5305	19.9793	19.9782	19.9791	16.4111	16.4114	16.4114	
	2	21.8161	21.8145	21.8126	20.3629	20.3617	20.3635	17.0602	17.0593	17.0585	
	1	22.0965	22.0956	22.0928	20.7394	20.7398	20.7352	17.6855	17.6845	17.6828	
	0	22.3733	22.3729	22.3738	21.109	21.1075	21.1057	18.2894	18.2901	18.2867	
	-1	22.6465	22.6461	22.6338	21.4722	21.4702	21.4702	18.874	18.8738	18.8703	
	-2	22.9162	22.9163	22.9134	21.8292	21.8303	21.8257	19.441	19.4401	19.4401	
	-3	23.1826	23.1813	23.1794	22.1803	22.1812	22.1756	19.992	19.9898	19.9880	

Table 2 represents the interactions between temperature, non-homogeneity, non-local parameter, and end support conditions on the non-dimensional natural frequency of the nanobeam. The first three non-dimensional frequencies have been calculated. In this table, two different values for the nonlocal parameter ($\psi=0$, $\psi=0.2$) are considered,

wherein the thermal force (P) and non-homogeneity (c) vary from -2 to two. The results have been presented for various end boundary conditions, such as simple–simple, clamped–clamped, clamped–simple, and clamped–free. The results prove that the increasing nonlocal parameter decreases the natural frequencies for all modes. The same result is obtained for an increase in temperature. On the other hand, the effects of increasing temperature and increasing nonlocal parameter are the same.

Table 2. Interaction between temperature, non-homogeneity, nonlocal parameter, and end support conditions.

End Conditions	p	$\mathbf{c}$	$\psi=0$			$\psi=0.2$			
			Mode 1	Mode 2	Mode 3	Mode 1	Mode 2	Mode 3	
Simple–Simple	-2	-2	Present	10.3279	40.7811	90.2939	8.8672	26.3066	43.8417
			[1]	10.3271	40.7773	90.3672	8.9497	26.4121	43.9336
		0	Present	10.8244	40.4667	89.8424	9.4652	26.1393	43.7195
			[1]	10.8252	40.4648	89.8203	9.4653	26.1387	43.7070
		2	Present	10.5129	40.8305	90.3142	9.0167	26.351	43.8625
			[1]	10.5127	40.8320	90.3984	9.1021	26.4551	43.9570
	0	-2	Present	9.4872	39.8522	89.3248	8.1079	24.9781	41.9203
			[1]	9.48682	39.8555	89.3984	8.1079	24.9785	41.9179
		0	Present	9.8696	39.4791	88.8482	8.3569	24.5827	41.6376
			[1]	9.8701	39.4805	88.8203	8.3569	24.5839	41.6289
		2	Present	9.4872	39.8522	89.3248	8.1079	24.9781	41.9203
			[1]	9.4868	39.8555	89.3984	8.1079	24.9785	41.9179
	2	-2	Present	8.5809	38.9007	88.3447	7.2861	23.5729	39.9057
			[1]	8.5805	38.9004	88.4297	7.1704	23.4550	39.8008
		0	Present	8.8142	38.4661	87.8426	7.0792	22.9204	39.4458
			[1]	8.8139	38.4668	87.8203	7.0796	22.9199	39.4336
		2	Present	8.3351	38.8497	88.3242	7.085	23.5253	39.8835
			[1]	8.3354	38.8496	88.3984	6.9741	23.4082	39.7773
End Conditions	p	$\mathbf{c}$	$\psi=0$			$\psi=0.2$			
			Mode 1	Mode 2	Mode 3	Mode 1	Mode 2	Mode 3	
Clamped–Clamped	-2	-2	Present	23.3832	63.1248	122.178	20.0715	38.5603	57.4924
			[1]	23.3847	63.1211	122.507	20.1465	38.6425	57.3867
		0	Present	22.9162	62.4294	121.918	19.441	38.3076	57.2176
			[1]	22.9160	62.4179	121.710	19.4404	38.2832	57.0117
		2	Present	23.4663	63.1564	122.187	20.1226	38.5977	57.5033
			[1]	23.4667	63.1523	122.523	20.2012	38.6816	57.4023
	0	-2	Present	22.9377	62.4241	121.375	19.0739	36.8686	55.1108
			[1]	22.9394	62.4258	121.726	19.0732	36.8497	54.9414
		0	Present	22.3733	61.6883	121.106	18.2894	36.4484	54.7272
			[1]	22.3730	61.6757	120.9141	18.2900	36.4238	54.5195
		2	Present	22.9377	62.4241	121.375	19.0739	36.8686	55.1108
			[1]	22.9394	62.4258	121.726	19.0732	36.8496	54.9570
	2	-2	Present	22.4875	61.7145	120.565	18.0261	35.0941	52.6212
			[1]	22.4863	61.7148	120.929	17.9365	34.9668	52.3789
		0	Present	21.8161	60.9375	120.289	17.0602	34.4889	52.118
			[1]	21.8144	60.9179	120.085	17.0596	34.4629	51.9258
		2	Present	22.3959	61.6825	120.556	17.9645	35.0542	52.6097
			[1]	22.3965	61.6836	120.9141	17.8779	34.9238	52.3711

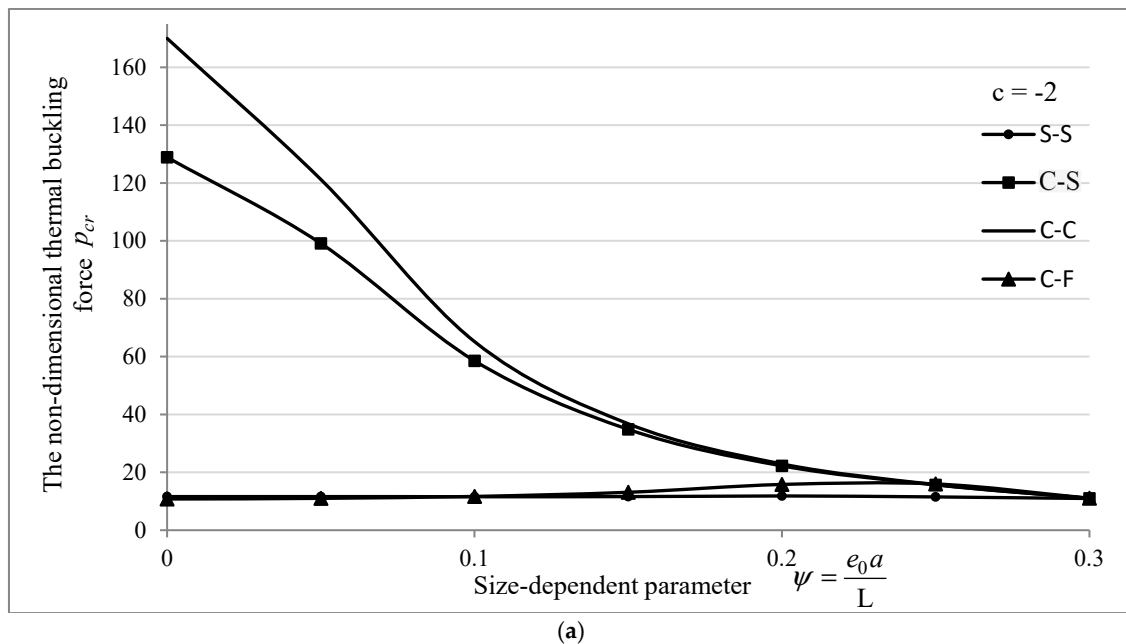
End Conditions	p	C	$\psi=0$			$\psi=0.2$		
			Mode 1	Mode 2	Mode 3	Mode 1	Mode 2	Mode 3
Clamped-Simple	-2	-2	Present	18.2396	53.3	107.577	15.8053	33.4217
			[1]	18.2393	53.3008	107.804	15.9443	33.5566
		0	Present	16.1469	50.819	105.278	13.8329	32.0247
			[1]	16.1475	50.8164	105.1484	13.8330	32.0215
		2	Present	14.2174	49.4018	103.8196	12.0371	31.1790
			[1]	14.2178	49.4023	103.8516	12.0615	31.2363
	0	-2	Present	17.7203	52.5272	106.713	14.9359	31.8917
			[1]	17.7197	52.5273	106.9453	14.9365	31.8886
		0	Present	15.4182	49.968	104.383	12.7458	30.3398
			[1]	15.4189	49.9648	104.2422	12.7451	30.3379
		2	Present	13.3439	48.5268	102.9106	11.108	29.6771
			[1]	13.3447	48.5273	102.9453	11.1084	29.6777
	2	-2	Present	17.1917	51.7421	105.840	14.0198	30.2831
			[1]	17.1924	51.7383	106.0859	13.8545	30.1269
		0	Present	14.6517	49.102	103.479	11.5565	28.5557
			[1]	14.6514	49.0977	103.3359	11.5557	28.5527
		2	Present	12.4045	47.6359	101.9936	10.0939	28.0953
			[1]	12.4053	47.6367	102.0234	10.0654	28.0332
Clamped-Free	-2	-2	Present	5.51088	26.923	67.0065	5.3870	20.8579
			[1]	5.51123	26.923	67.0039	5.4624	21.0410
		0	Present	2.63488	22.5347	62.4216	2.8520	18.7399
			[1]	2.63501	22.5332	62.4023	2.8523	18.7393
		2	Present	-	18.7996	59.1373	1.3712	16.3861
			[1]	-	18.7998	59.1367	1.2200	16.3994
	0	-2	Present	6.2626	26.5836	66.3764	5.8515	19.9513
			[1]	6.2622	26.5839	66.3712	5.8511	19.9512
		0	Present	3.5160	22.0345	61.7151	3.5794	17.576
			[1]	3.5158	22.0332	61.6992	3.5793	17.575
		2	Present	1.8405	18.1721	58.3915	2.4652	15.2318
			[1]	1.8405	18.1729	58.3867	2.4651	15.2314
	2	-2	Present	7.0054	26.2272	65.7388	6.3285	18.9899
			[1]	7.0054	26.2285	65.7383	6.2554	18.7822
		0	Present	4.2921	21.5078	60.9997	4.2810	16.2998
			[1]	4.2922	21.5097	60.9805	4.2810	16.2998
		2	Present	2.7431	17.5078	57.6359	3.3039	13.9582
			[1]	2.7434	17.5068	57.6367	3.4187	13.9228

The changes in parameter p from -2 to zero and then to two correspond to a reversal of the axial force from compression to tension. While the compressive axial force reduces the structural stiffness and consequently decreases the natural frequency, the tensile axial force produces a stiffening effect and enhances the vibration characteristics of the structure. Accordingly, the natural frequency increases. For example, in Table 2, for the “C–F” beam with $\psi = 0.2$ and $c = -2$, the first frequency increases from approximately 5.3870 to 5.8515 and then to 6.3285 as “ p ” changes from -2 to zero and two. A similar trend is observed for $c = 2$. Comparable trends regarding the variation in the buckling load with the size-effect parameter are reported in Refs. [37,38], wherein the modified couple stress theory and the nonlocal third-order shear deformation beam the-

ory have been utilized to determine the thermal buckling load. Regarding the limiting case ($p = 0$, $c = 0$ and $\psi = 0$) in Table 2, it can be seen that the relative error percentage is higher for higher modes than for lower modes. This can be attributed to the fact that shear deformations become more significant in higher modes, since the present method uses the Bernoulli beam theory, which does not account for the effects of shear deformations. As can be seen, the relative error reaches a maximum of 3.8% ($P = -2$, $c = 2$, $\psi = 0.2$), which is entirely acceptable. In the source, this error can be attributed to the power series considered in Ref. [1]. In the method presented in the present work, only the vibration mode shape function is approximated by a power series, while the remaining derivatives of the vibration mode shape function are computed exactly and analytically and are used in the integrations. In Ref. [1], however, the vibration mode shape function and its first- through fourth-order derivatives are approximated by a power series, which introduces a larger percentage error into the calculations. This very point can be regarded as an advantage of the Fredholm integral method over the power series method.

5.2. Thermal Buckling Force

In this section, we investigate the effects of the size-dependent parameter on the thermal buckling force of the Axially Functionally Graded (AFG) nanobeam. Figure 3a–c present variations in the thermal buckling force with the size-dependent parameter. The results have been obtained for three gradient indexes $c = -2, 0, 2$. The results from all four boundary conditions have been combined to compare them. It has been assumed that the size-dependent parameter varies from zero to 0.3.



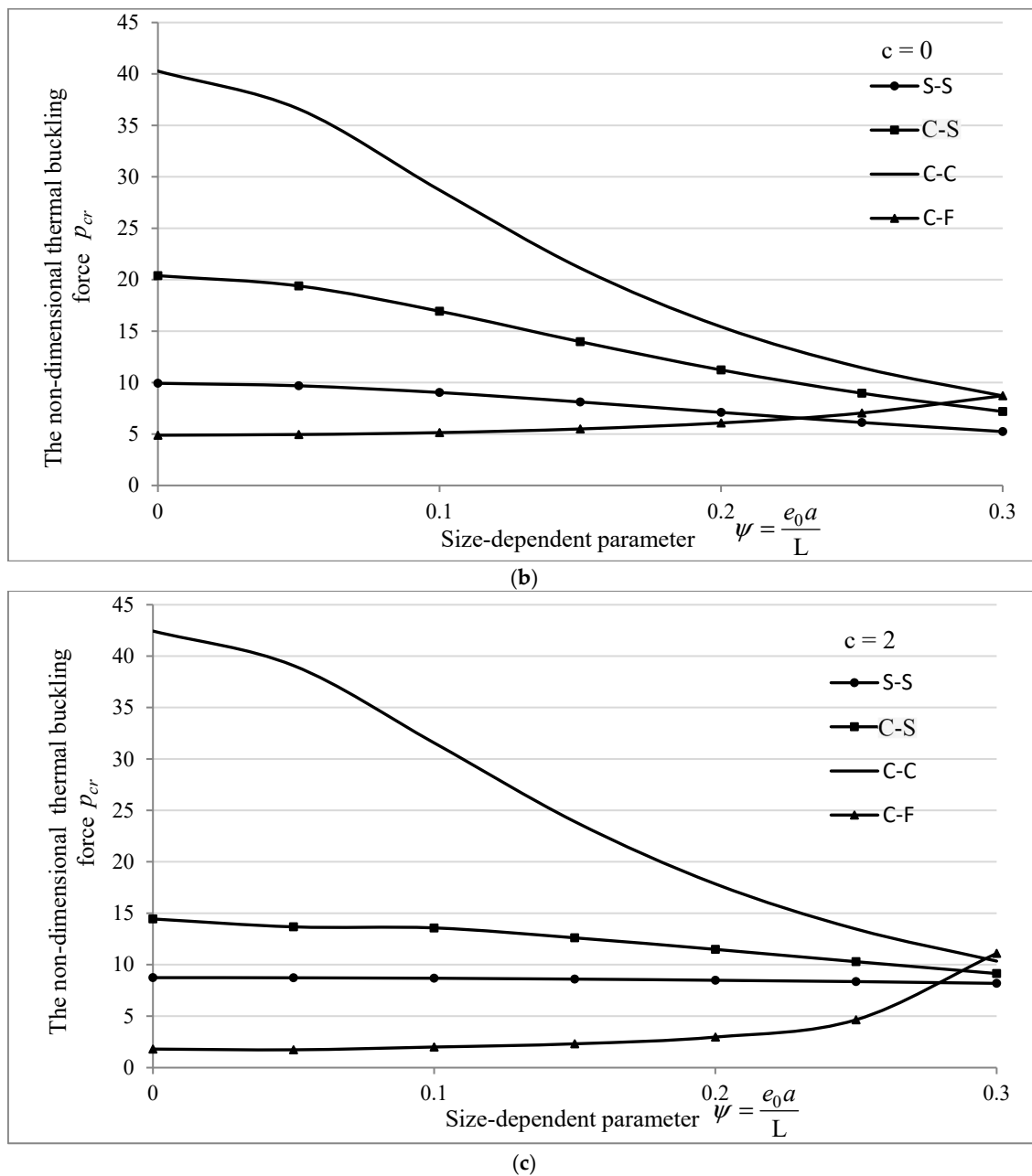


Figure 3. (a) The effects of the size-dependent parameter on the thermal buckling force ($c = -2$); (b) the effects of size-dependent parameter on the thermal buckling force ($c = 0$); (c) the effects of the size-dependent parameter on the thermal buckling force ($c = 2$).

5.3. Effects of Thermal Axial Force on the Natural Frequencies

Figures 4 and 5 present the variations in the first three natural frequencies with the thermal axial force. These results have been obtained with a fixed size-dependent parameter of $\psi = 0.2$ and two gradient indexes of $c = -2$ and two. It has been assumed that the thermal axial force varies from -2 to two. In these figures, the results from all four boundary conditions have been combined to compare them.

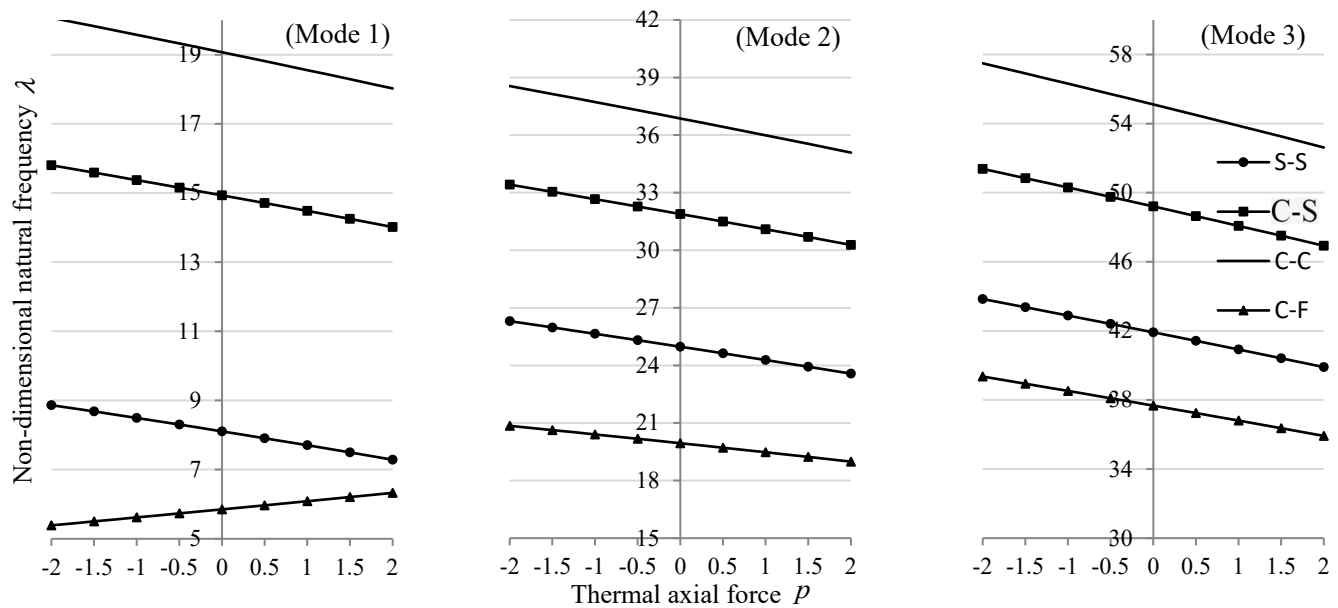


Figure 4. The variations in the first three natural frequencies with the thermal axial force ($c=-2$, $\psi=0.2$).

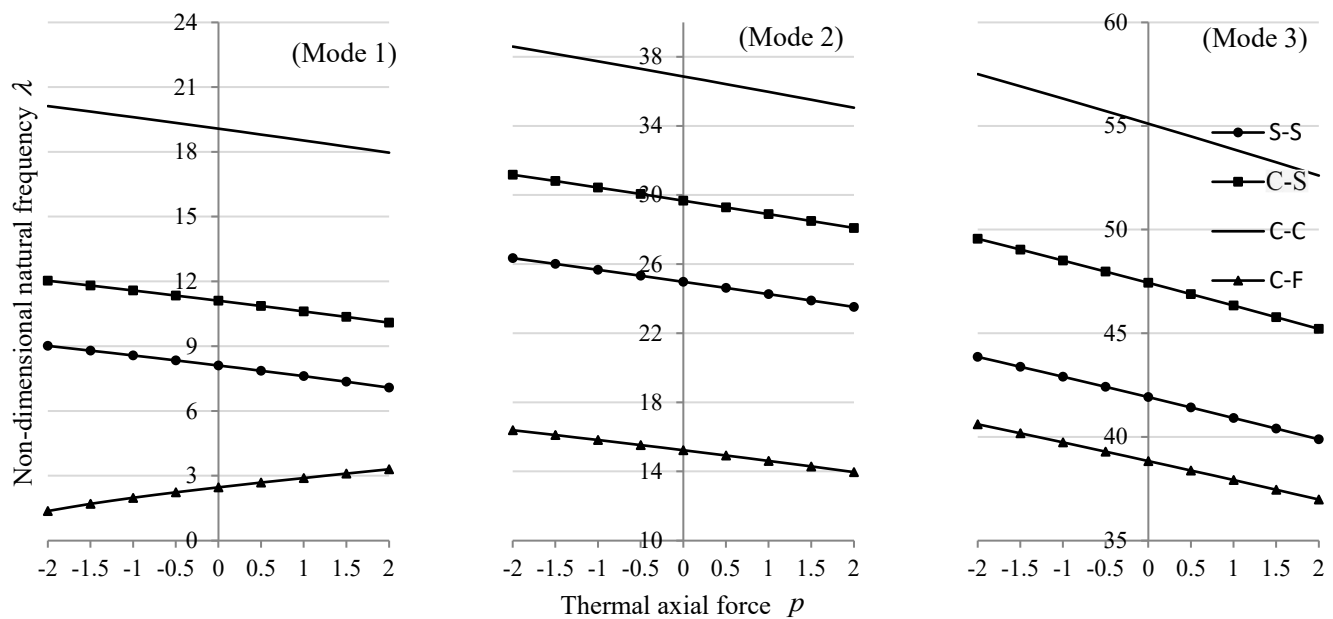
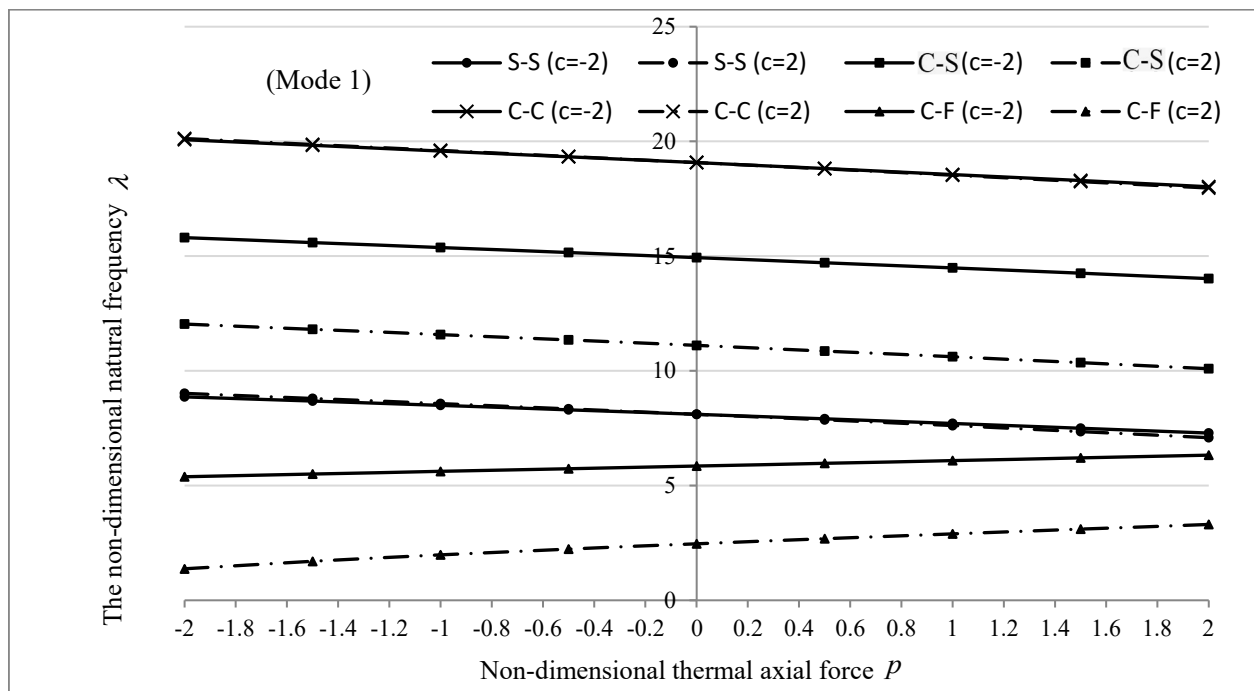
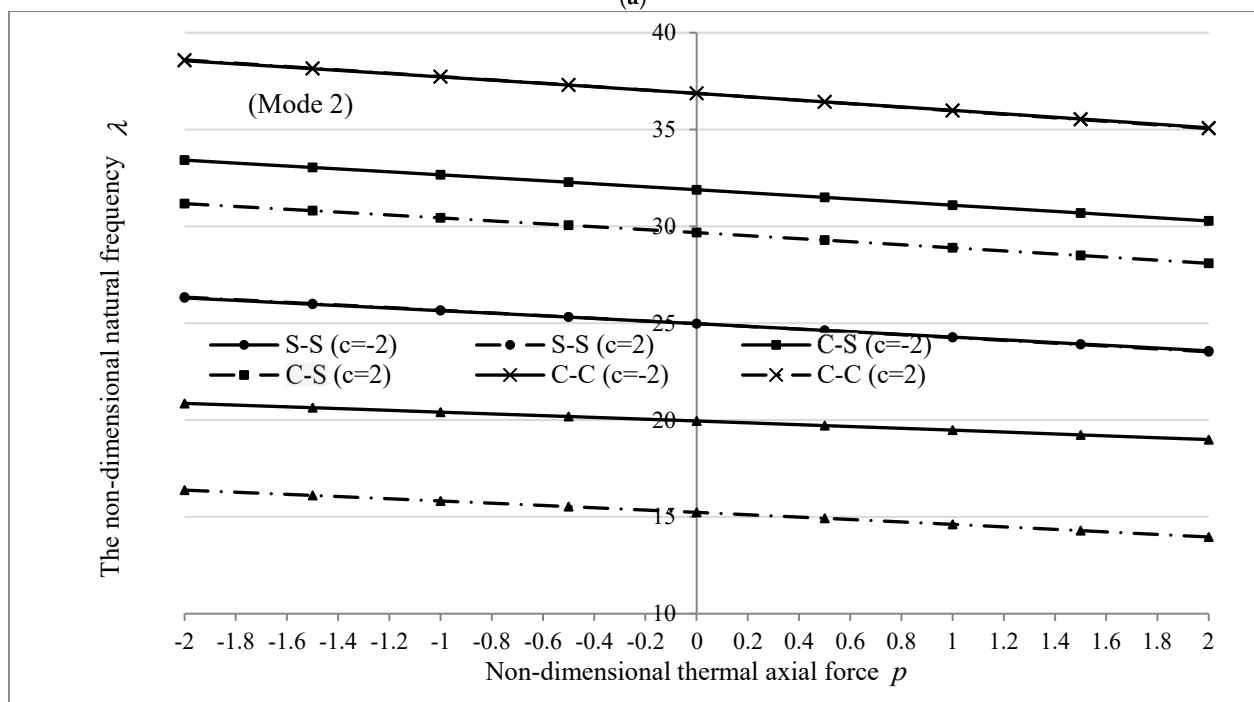


Figure 5. The variations in the first three natural frequencies with the thermal axial force ($c = 2$, $\psi=0.2$).

To investigate the effect of changing the gradient index from -2 to two, the variations in the natural frequencies with respect to the thermal axial force for the two gradient index values $c = -2$ and $c = 2$ are simultaneously presented in Figure 6a–c. The results have been obtained for the first three modes. As can be seen, for some boundary conditions, the results obtained for $c = -2$ and $c = 2$ are consistent.



(a)



(b)

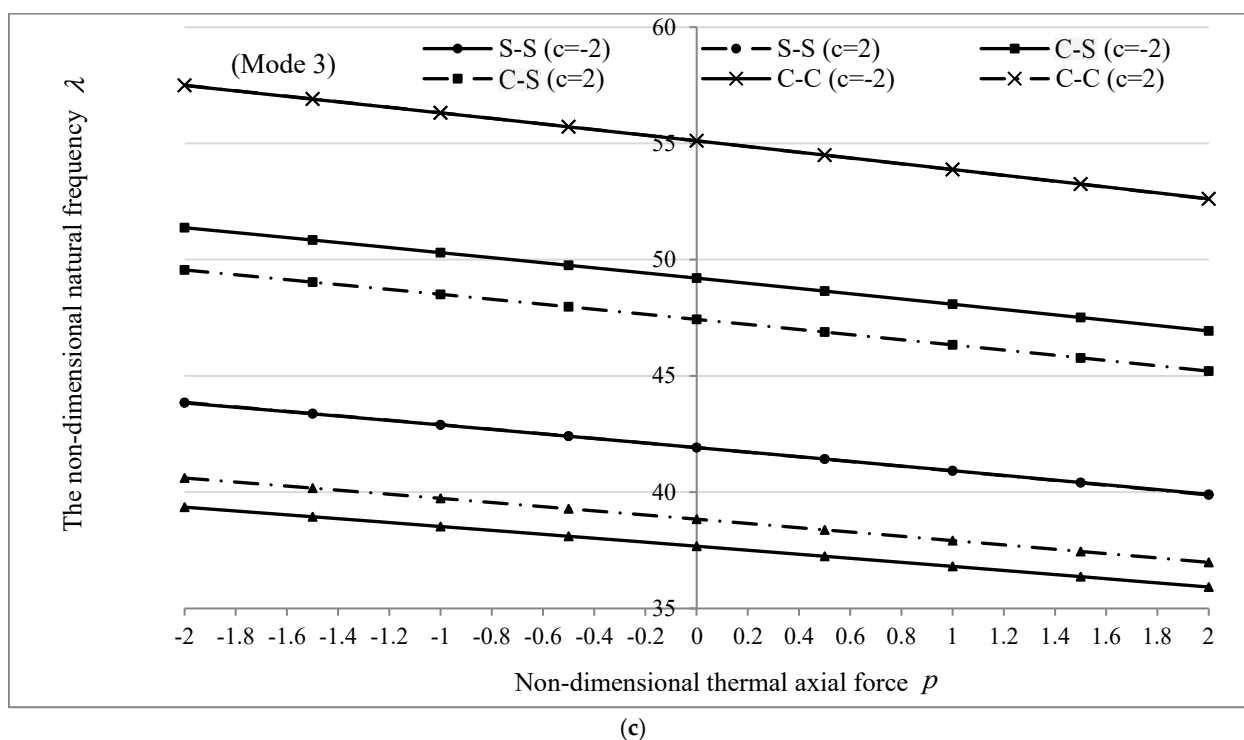


Figure 6. (a) The effect of changing the gradient index from -2 to two (Mode 1); (b) the effect of changing the gradient index from -2 to two (Mode 2); (c) the effect of changing the gradient index from -2 to two (Mode 3).

To assess the convergence rate of the proposed method, the first five non-dimensional natural frequencies of a beam with clamped–clamped boundary conditions were computed for cases $\{P = -2, c = -2, \psi = 0.2\}$ and $\{P = 2, c = 2, \psi = 0.2\}$, when considering the increasing values of “ n ”. Also, the non-dimensional thermal buckling force p_{cr} was calculated for case $\{c = 2, \psi = 0.2\}$. The results are summarized in Table 3.

Table 3. The convergence rate of the proposed method.

		The non-dimensional natural frequencies λ					
		Mode number	1	2	3	4	5
Clamped–Clamped	$P = -2, c = -2, \psi = 0.2$	R = 1	28.14817	96.3726	N.A	N.A	N.A
		R = 3	20.24016	39.8420	281.1011	1072.99	N.A
		R = 5	20.07165	38.5315	62.5648	86.0600	1738.952
		R = 7	20.07151	38.5401	57.5180	75.4445	113.9392
		R = 10	20.07151	38.5405	57.3133	74.7334	92.44669
		R = 12	20.07151	38.5405	57.3133	74.7098	92.33414
		The non-dimensional natural frequencies λ					
	$P = 2, c = 2, \psi = 0.2$	Mode number	1	2	3	4	5
		R = 1	25.9588	88.9539	N.A	N.A	N.A
		R = 3	18.1073	36.2420	259.5136	990.34	N.A
		R = 5	17.9645	35.0229	57.3340	79.1085	1605.01
		R = 7	17.9644	35.0355	52.6371	69.2568	104.902
		R = 10	17.9644	35.0359	52.4470	68.5994	84.9992
		R = 12	17.9644	35.0359	52.4470	68.5775	84.8954
The thermal buckling force p_{cr}							
R	1	2	3	4	5		

p_{cr}	25.6609	17.5590	17.8184	17.7973	17.8485
R	6	7	8	9	10
p_{cr}	17.8497	17.85	17.85	17.85	17.85

The results in Table 3 demonstrate the rapid convergence of the proposed method and its ability to calculate the frequencies of higher modes as the parameter “n” increases. The values of the size-dependent parameter considered in the numerical examples were adopted from the benchmark studies used for validation and are consistent with those commonly employed in the literature on Eringen’s differential nonlocal elasticity. This range includes the local elasticity limit and the representative levels of nonlocal interactions, thereby enabling a comprehensive assessment of the influence of size effects on the structural response.

6. Discussion on the Results

Verification of Results (Table 1): The proposed relations for natural frequency calculations align well with References [1,36], confirming the accuracy of the methodology. Increasing the nonlocal parameter reduces the first natural frequency due to size-dependent effects. Decreasing the temperature increases the first natural frequency. Boundary conditions (simple–simple vs. clamped–clamped) significantly influence frequency values (e.g., simple–simple: $\lambda_1 \approx 9.87$; clamped–clamped: $\lambda_1 \approx 22.37$ at $p = 0$).

Parameter Interactions (Table 2): Increasing the nonlocal parameter reduces natural frequencies across all modes (Modes 1–3) and boundary conditions. Increasing the temperature decreases the natural frequencies, mirroring the effect of increasing the nonlocal parameter. Variations in material gradient (e.g., $c = -2$ vs. $c = 2$) lead to distinct frequency responses, particularly in higher modes (Modes 2–3). Clamped–clamped beams exhibit the highest natural frequencies, while clamped–free beams show the lowest. As an example, for $p = -2$, Mode 1 frequencies range from $\lambda_1 = 5.51$ (clamped–free) to $\lambda_1 = 23.38$ (clamped–clamped).

Thermal Buckling Analysis: Increasing the parameter ($0 \leq \psi \leq 0.3$) reduces the thermal buckling force, with gradient index c modulating this effect. For $c = -2$, the buckling force decreases sharply with ψ . For $c = 2$, the reduction is more gradual. Clamped–clamped beams exhibit the highest buckling resistance, while clamped–free beams are most susceptible. The influence of the nonlocal parameter on the thermal buckling load depends on the boundary conditions. For the S–S, C–C, and C–S beams, increasing the nonlocal parameter generally decreases the thermal buckling load, indicating a reduction in thermal buckling resistance due to the nonlocal softening effect. In contrast, the C–F beam exhibits a different trend over part of the investigated parameter range, indicating that the sensitivity of the thermal buckling load to the nonlocal parameter is strongly influenced by the support conditions. Therefore, no universal conclusion regarding the effect of the nonlocal parameter can be drawn without considering the boundary conditions (Figure 3a–c).

Natural Frequency vs. Thermal Axial Force: Increasing the compressive thermal force (positive values) reduces the natural frequencies, while the tensile force (negative values) increases them. For $c = -2$, Mode 1’s frequency decreases by $\sim 17.8\%$ as thermal force shifts from -2 to $+2$ for simple–simple beam (Figure 4). Switching from $c = -2$ to $c = 2$ alters the frequency trends, with some boundary conditions showing consistent results despite gradient changes (e.g., clamped–clamped; Figure 4 vs. Figure 5). The analysis of natural frequency variations under thermal axial force (Figure 6a–c) reveals that certain boundary conditions (e.g., clamped–clamped) demonstrate consistent frequency responses across gradient indexes—indicating material gradation insensitiv-

ty—whereas others (e.g., clamped–free) show significant divergence. Although the numerical results are presented in a dimensionless form to facilitate comparison with existing benchmark solutions, the corresponding representative material properties and geometric dimensions adopted in the numerical examples are identical to those reported in Reference [1]. These reference values enable the physical interpretation and reproducibility of the results while preserving the generality of the proposed dimensionless formulation. For the sake of brevity, these data have not been repeated in the present paper.

The present study advances the analysis of functionally graded (FG) nanobeams by introducing a unified thermo-elastic framework that integrates Eringen’s nonlocal elasticity with a Fredholm integral transformation to investigate both free vibration and thermal buckling within a single semi-analytical formulation. Unlike Li et al. [39], who developed a viscoelastic integral nonlocal strain gradient theory (VINSGT) for curved Timoshenko nanobeams resting on viscoelastic foundations, the present work focuses on Euler–Bernoulli FG nanobeams subjected to thermal loading and material gradation. While both studies adopt integral nonlocal formulations to accurately capture nanoscale phenomena, their objectives differ substantially. Li et al. [39] emphasized damping characteristics through Kelvin–Voigt viscoelasticity, generalized differential quadrature discretization, and complex eigenvalue analysis, enabling the prediction of time-dependent vibration attenuation in M/NEMS devices. In contrast, the present study neglects viscoelastic damping but develops a semi-analytical Fredholm transformation that converts the governing differential equation into a weak-form integral equation solved through the Galerkin weighted residual method. Consequently, the proposed methodology offers explicit semi-analytical insight into the interactions between thermal loading, size effects, and material gradation, whereas Li et al. [39] primarily addressed dynamic dissipation mechanisms and foundation-induced damping. Moreover, the current study extends applicability to thermal instability by determining critical thermal buckling loads, an aspect beyond the scope of Li et al. [39].

Compared with Yang et al. [40], both investigations incorporate advanced nonclassical continuum theories to improve nanoscale structural predictions, yet they address fundamentally different multiphysics problems. Yang et al. [40] formulated a higher-order gradient elasticity model coupled with flexomagnetism and surface elasticity to investigate geometrically nonlinear bending of piezomagnetic Timoshenko nanobeams under large deformation using a meshless finite block method. Conversely, the present study assumes small deformation within the Euler–Bernoulli framework but explicitly integrates thermal expansion, exponential material gradation, and nonlocal elasticity into a mathematically tractable Fredholm integral formulation. Therefore, while Yang et al. [40] concentrated on nonlinear magneto-mechanical coupling and surface effects essential for smart flexomagnetic devices, the present study provides an efficient semi-analytical solution for thermo-mechanical stability and vibration. This distinction demonstrates that the proposed formulation complements rather than competes with higher-order multiphysics models by offering a computationally efficient semi-analytical alternative for thermally loaded FG nanobeams.

A comparison with Luo et al. [41] further highlights the methodological originality of the present work. Luo et al. [41] modeled defective FG nanobeams by treating material properties as interval variables and proposed interval spline finite point methods combined with improved Taylor expansion algorithms to quantify uncertainty arising from defects. Their primary contribution lies in uncertainty quantification and interval mechanics. In contrast, the present study assumes deterministic exponential material gradation and instead concentrates on deriving a semi-analytical weak-form integral representation of the governing equations through successive Fredholm integrations.

Although both studies employ nonlocal elasticity and investigate FG nanobeams, their scientific emphases differ substantially. Luo et al. [41] addressed uncertainty caused by manufacturing imperfections, whereas the present study elucidates the coupled influence of thermal loading, nonlocality, and boundary conditions on vibration frequencies and thermal buckling resistance. Consequently, the present formulation establishes a semi-analytical benchmark that could potentially serve as the deterministic foundation for future interval-based or stochastic extensions.

Relative to Li et al. [42], who proposed a mixed finite element (MIF5) formulation for rotating FG nanobeams based on the Hellinger–Reissner variational principle, the present study adopts an entirely different computational philosophy. Li et al. [42] focused on overcoming convergence limitations and nonlocal paradoxes in differential formulations through a mixed finite element strategy, demonstrating superior numerical convergence for rotating nanobeams. In contrast, the present work avoids conventional finite element discretization altogether by transforming the governing equations into Fredholm integral equations before applying a Galerkin approximation. This semi-analytical integral framework naturally incorporates boundary conditions and enables the simultaneous determination of natural frequencies and thermal buckling loads without developing specialized finite elements. Furthermore, whereas Li et al. [42] investigated rotational inertia and centrifugal effects, the present study emphasizes thermo-elastic instability, demonstrating that increasing the nonlocal parameters consistently reduces both natural frequencies and thermal buckling resistance, all while boundary conditions strongly influence the structural performance.

The present work also exhibits strong conceptual consistency with Song et al. [43], whose theoretical contribution clarified the implicit existence of constitutive boundary conditions (CBCs) and constitutive interface conditions (CICs) within integral stress-driven nonlocal formulations. Both studies emphasize the mathematical superiority of integral nonlocal approaches over purely differential formulations. However, whereas Song et al. [43] focused on the theoretical interpretation of CBCs and CICs in stress-driven nonlocal elasticity, the present study develops a practical semi-analytical framework in which successive Fredholm integrations naturally introduce integration constants corresponding to shear force, bending moment, slope, and displacement, with boundary conditions systematically enforced throughout the transformation process. Thus, the current methodology provides an applied realization of the theoretical advantages highlighted by Song et al. [43], demonstrating how integral formulations facilitate mathematically consistent implementation of boundary conditions while producing accurate predictions of thermo-mechanical behavior. Collectively, these comparisons demonstrate that the present study contributes a distinctive semi-analytical framework integrating thermal effects, material gradation, nonlocal elasticity, and Fredholm integral transformation, thereby extending the semi-analytical toolbox available for the design and optimization of advanced functionally graded nanostructures operating under coupled thermal and size-dependent conditions.

7. Conclusions

This analysis revealed the critical interplay between nonlocal effects, material gradation, and boundary conditions in functionally graded nanobeams, providing essential insights for designing nanostructures with tailored thermo-mechanical properties. The findings have established design guidelines for nano-mechanical systems operating in thermal environments, emphasizing gradient index optimization based on boundary constraints. A unified framework for analyzing the functionally graded Euler–Bernoulli nanobeams through coupled thermal buckling and free vibration analysis was established. The proposed nonlocal thermo-elastic Fredholm integral solution integrated ex-

ponential material gradation in elastic properties and density with Eringen's size-dependent nonlocal elasticity. Thermal effects were incorporated through an equivalent axial force formulation. The proposed formulation was validated through close agreement with existing studies, showing that nonlocal effects, temperature, and boundary conditions strongly influenced natural frequencies. Material gradation primarily affected higher modes, while increasing the nonlocal parameters reduced natural frequencies. No universal conclusion regarding the effect of the nonlocal parameter on the thermal buckling resistance can be drawn without considering the boundary conditions. For example, the C–F beam exhibited a different trend over part of the investigated parameter range. The proposed formulation was developed within the framework of Eringen's differential nonlocal Euler–Bernoulli beam theory and is therefore most appropriate for slender nanobeams, where transverse shear deformation and rotary inertia can be neglected. In addition, the material properties were assumed to follow an exponential gradation, while the coefficient of thermal expansion was taken as constant to enable the development of the analytical Fredholm integral solution. Although these assumptions are commonly adopted in analytical studies, they may not capture the full complexity of practical functionally graded nanomaterials with temperature-dependent properties or more general gradation profiles. Furthermore, the present work is restricted to linear free vibration and thermal buckling analyses. Extending the proposed Fredholm integral framework to higher-order beam theories, nonlinear analyses, non-uniform thermal environments, and more general material models represents an important direction for future research.

Author Contributions: Conceptualization, M.M. and M.G.; methodology, M.M.; software, M.M.; validation, M.G., B.L. and K.B.; formal analysis, M.M. and M.G.; investigation, M.M. and M.G.; resources, M.M. and M.G.; data curation, M.M.; writing—original draft preparation, M.M., K.B. and M.G.; writing—review and editing, B.L. and K.B.; visualization, M.M. and M.G.; supervision, M.M., B.L. and K.B.; project administration, M.G.; funding acquisition, B.L. All authors have read and agreed to the published version of the manuscript.

Funding: This research was funded by internal funds from the Gdynia Maritime University, under the project number WM/2026/PZ/02.

Data Availability Statement: The raw data supporting the conclusions of this article will be made available by the authors upon request.

Conflicts of Interest: The authors declare no conflicts of interest.

Appendix A

Thermal axial force

The total axial strain is calculated as follows:

$$\varepsilon(x) = \varepsilon_m(x) - \alpha\Delta T \quad (\text{A1})$$

In Equation (A1), $\varepsilon_m(x)$ and $\alpha\Delta T$ denote mechanical strain and thermal strain, respectively. According to classical mechanics of materials relations, the axial force in the presence of temperature changes is calculated as follows:

$$N(x) = AE(x)[\varepsilon_m(x) - \alpha\Delta T] \quad (\text{A2})$$

For an isolated beam without a distributed axial load, axial equilibrium generally requires:

$$\frac{dN}{dx} = 0 \rightarrow N(x) = N_t = \text{constant} \quad (\text{A3})$$

The global compatibility conditions for the beam require that:

$$\int_0^L \varepsilon_m(x) dx = 0 \quad (\text{A4})$$

Using Equation (A2), $\varepsilon_m(x)$ is introduced into Equation (A4). The result is:

$$\int_0^L \left[\frac{N_t}{AE(x)} + \alpha \Delta T \right] dx = 0 \quad (\text{A5})$$

Using Equation (A5), the thermal axial force is obtained as follows:

$$N_t = - \frac{A \alpha \Delta T L}{\int_0^L \frac{dx}{E(x)}} \quad (\text{A6})$$

Functions $f_2(\xi, s)$ for each boundary condition

The following functions and parameters can be used to calculate the function $f_2(\xi, s)$ for each boundary condition:

$$\begin{aligned} \text{Simple-Simple: } & \begin{cases} \beta = \frac{24\alpha_1 - 12\alpha_1^2}{4\alpha_1^2 - 6\alpha_1^2 + 24} \\ g_3(s) = \frac{2}{2-\alpha_1} g_1(s) + \frac{2\beta\alpha_1}{2-\alpha_1} g_2(s) \\ g_2(s) = f_1(1, s) - \frac{2}{12-6\alpha_1} g_1(s) \\ g_1(s) = -\frac{\alpha_1\alpha_4}{2}(1-s)^2 + (\alpha_4 - \alpha_1\alpha_3)(1-s) - \alpha_1\alpha_2 + \alpha_3 \end{cases} \\ \text{Clamped-Clamped: } & \begin{cases} g_1(s) = -12f_1(1, s) + 6\alpha_2 + 6\alpha_3(1-s) + 3\alpha_4(1-s)^2 \\ g_2(s) = 6f_1(1, s) - 2\alpha_2 - 2\alpha_3(1-s) - \alpha_4(1-s)^2 \end{cases} \\ \text{Clamped-Simple: } & \begin{cases} g_3(s) = \left(\frac{36-18\alpha_1}{\alpha_1-4} + 6 \right) f_1(1, s) - \left(\frac{6}{\alpha_1-4} \right) g_1(s) \\ g_2(s) = \left(\frac{2}{\alpha_1-4} \right) g_1(s) + \left(\frac{6\alpha_1-12}{\alpha_1-4} \right) f_1(1, s) \\ g_1(s) = \alpha_3 + \alpha_4(1-s) - \alpha_1\alpha_2 - \alpha_1\alpha_3(1-s) - \frac{\alpha_1\alpha_4}{2}(1-s)^2 \end{cases} \end{aligned}$$

$$\begin{aligned}
 & \left. \begin{aligned}
 g_1(s) &= \alpha_1^2 f_1(1, s) - \alpha_1 \alpha_2 - \alpha_1 \alpha_3 (1-s) - \frac{\alpha_1 \alpha_4}{2} (1-s)^2 - \alpha_2 f_1(1, s) + \alpha_3 + \alpha_4 (1-s) \\
 g_2(s) &= \alpha_1 \alpha_2 f_1(1, s) - \alpha_2^2 - \alpha_2 \alpha_3 (1-s) - \frac{\alpha_2 \alpha_4}{2} (1-s)^2 - \alpha_3 f_1(1, s) + \alpha_4 \\
 g_3(s) &= \left(\frac{\beta_1 \beta_2 \beta_3}{\beta_4 \beta_1^3 - \beta_1^2 \beta_2 \beta_3} + \frac{1}{\beta_1} \right) g_1(s) - \left(\frac{\beta_1 \beta_2}{\beta_4 \beta_1^2 - \beta_1 \beta_2 \beta_3} \right) g_2(s) \\
 g_4(s) &= \left(\frac{\beta_1}{\beta_4 \beta_1 - \beta_2 \beta_3} \right) g_2(s) - \left(\frac{\beta_1 \beta_3}{\beta_4 \beta_1^2 - \beta_1 \beta_2 \beta_3} \right) g_1(s) \\
 \beta_1 &= -\frac{\alpha_1}{2} + \frac{\alpha_1^2}{6} - \frac{\alpha_2}{6} + 1 \\
 \beta_2 &= -\alpha_1 + \frac{\alpha_1^2}{2} - \frac{\alpha_2}{2} + 1 \\
 \beta_3 &= -\frac{\alpha_2}{2} + \frac{\alpha_1 \alpha_2}{6} - \frac{\alpha_3}{6} + 1 \\
 \beta_4 &= -\alpha_2 + \frac{\alpha_1 \alpha_2}{2} - \frac{\alpha_3}{2}
 \end{aligned} \right\} \text{Clamped-Free:}
 \end{aligned}$$

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