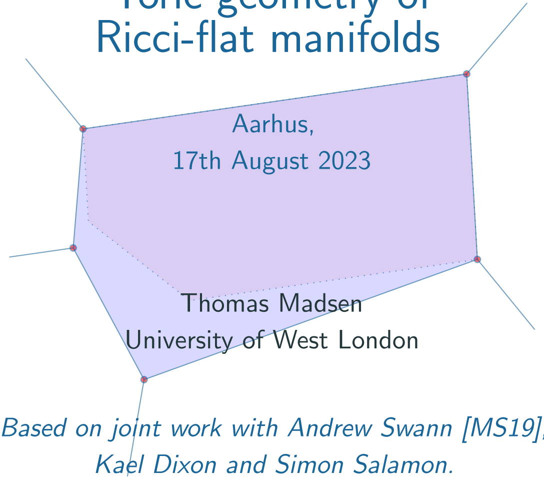
Toric geometry of Ricci-flat manifolds



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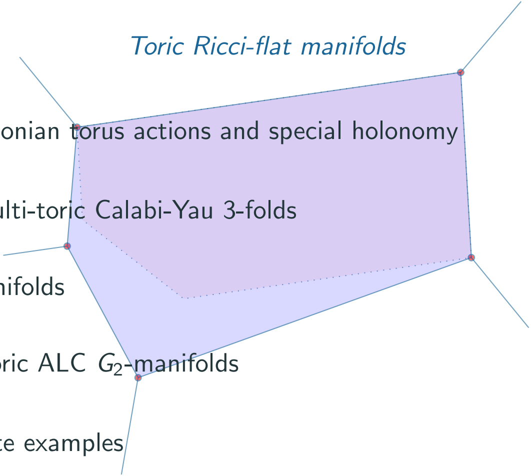
Based on joint work with Andrew Swann [MS19], Kael Dixon and Simon Salamon.

Toric Ricci-flat manifolds



Multi-Hamiltonian torus actions and special holonomy

Toric and multi-toric Calabi-Yau 3-folds

Toric G_2-manifolds

Producing toric ALC G_2-manifolds

New complete examples

Multi-Hamiltonian torus actions and special holonomy

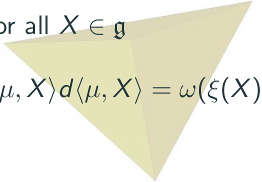
Toric symplectic geometry

(N^{2n}, ω) compact symplectic with effective Hamiltonian action of $G = T^n$. Have associated *moment map*

$$\mu \colon N \to \mathfrak{g}^* \cong \mathbb{R}^n$$

which is invariant and for all $X \in \mathfrak{g}$

$$\langle \mu, X \rangle d\langle \mu, X \rangle = \omega(\xi(X), \cdot).$$



- If $b_1(N) = 0$, then T^n a action preserving ω is Hamiltonian if and only if all orbits are isotropic.
- Codimension of generic orbit equals that of target space of μ.
- Stabiliser of any point is subtorus of dim $n - \operatorname{rank} d\mu$.
- μ identifies orbit space, N/G, with a convex polytope.

Riemannian setting: Ricci-flat special holonomy

4 types of *Ricci-flat* geomtries appear on Berger's list of special holonomy groups. Each is defined by one or more closed differential forms:

Name	Holonomy group	Dimension	Forms degree
Calabi-Yau	$\mathrm{SU}(n)$	$2n$	$2, n, n$
HyperKähler	$\mathrm{Sp}(n)$	$4n$	$2, 2, 2$
G_2	G_2	7	$3, 4$
Spin(7)	$\mathrm{Spin}(7)$	8	4

A symplectic manifold is a "closed form" geometry. When admitting torus symmetry, moment map techniques can be used to construct many examples and obtain classifications.

What about above geometries?

Note due to Ricci-flatness, torus symmetry will force us to look at (complete) non-compact spaces.

Multi-Hamiltonian actions [MS13]

N with closed $\alpha \in \Omega^{p+1}(N)$ preserved by action of Abelian G.

Action is multi-Hamiltonian if there is invariant $\nu\colon N \to \Lambda^p \mathfrak{g}^*$ s.t. for all $X_i \in \mathfrak{g}$

$$\langle \nu, X_1 \wedge \cdots \wedge X_p \rangle d\langle \nu, X_1 \wedge \cdots \wedge X_p \rangle = \alpha(\xi(X_1), \ldots, \xi(X_p), \,\cdot\,).$$

Our interest is $G = T^n$, acting effectively:

- Take $n \geqslant p$.
- If $b_1(N) = 0$, then T^n-action preserving α is multi-Hamiltonian if and only if α pulls back to zero on each orbit.

If we have k invariant closed forms $\alpha_i \in \Omega^{p_i+1}(N)$ with multi-moment maps ν_i, we form the product multi-moment map

$$\nu = (\nu_1, \ldots, \nu_k)\colon\ N \to \bigoplus_{i=1}^{k} \Lambda^{p_i} \mathfrak{g}^*\,.$$

Capturing orbit space with multi-moment maps

Let $N_0 \subset N$ be the open dense set where the torus G acts freely and let $q = \dim(N_0/G)$ be the co-dimension of generic orbits.

An interesting case is when the multi-moment map

$$\nu\colon\ N_0 \to \mathbb{R}^q$$

has full rank. Then ν locally exhibits N_0 as a principal G-bundle over $\mathcal{U} = \nu(N_0) \subset \mathbb{R}^q$.

For the Ricci-flat special holonomy geometries, the above requires:

Type	$\dim(N)$	$\deg \alpha_i$	G	q
Calabi-Yau	$2n$	$2, n, n$	T^{n-1}	$n+1$
HyperKähler	$4n$	$2, 2, 2$	T^n	$3n$
G_2	7	$3, 4$	T^3	4
Spin(7)	8	4	T^4	4

Toric and multi-toric Calabi-Yau 3-folds

Calabi-Yau 3-folds

This is B^6 with $\omega \in \Omega^2(B)$ and $\Psi = \psi + i\widehat{\psi} \in \Omega^3(B, \mathbb{C})$ pointwise linearly equivalent to

$$\omega_0 = \frac{i}{2}(d\ _1 \wedge d\bar{\ }_1 + d\ _2 \wedge d\bar{\ }_2 + d\ _3 \wedge d\bar{\ }_3) \in \Lambda^2(\mathbb{C}^3)^*$$

and

$$\Psi_0 = d\ _1 \wedge d\ _2 \wedge d\ _3 \in \Lambda^3(\mathbb{C}^3)^*.$$

The $\mathrm{GL}(6, \mathbb{R})$ stabiliser of ω_0 *and* Ψ_0 is $\mathrm{SU}(3) \leqslant \mathrm{SO}(6)$. In particular, (ω, Ψ) determines Riemannian metric h via:

$$-\frac{1}{3}h(X, Y)\,\omega^3 = (X \lrcorner \omega) \wedge (Y \lrcorner \psi) \wedge \psi.$$

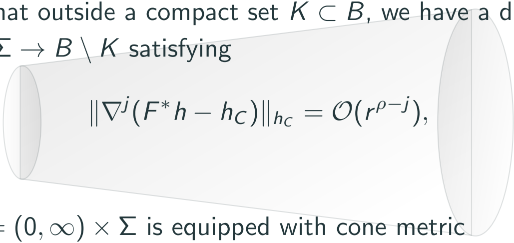
Holonomy of h is in $\mathrm{SU}(3)$ $\iff$ $d\omega = 0$ and $d\Psi = 0$.

AC Calabi-Yau 3-folds

In what follows, we will asssume (B, ω, Ψ) is *asymptotically conical* (of rate $\rho < 0$).

This means that outside a compact set $K \subset B$, we have a diffeomorphism $F\colon (0, \infty) \times \Sigma \to B \setminus K$ satisfying

$$\|\nabla^j (F^* h - h_C)\|_{h_C} = \mathcal{O}(r^{\rho - j}),$$



for all $j \geqslant 0$.

Here, $C(\Sigma) = (0, \infty) \times \Sigma$ is equipped with cone metric

$$h_C = dr^2 + r^2 g_\Sigma.$$

The link, Σ^5 is a so-called Sasaki-Einstein manifold.

Toric CY 3-folds

In traditional sense, B is called *toric* if it comes with an effective T^3-action preserving ω and complex structure.

In paticular, we get a (usual) moment map:

$$\mu = (\mu_1, \mu_2, \mu_3) \colon B^6 \to (\mathfrak{t}^3)^* \cong \mathbb{R}^3.$$

Lemma

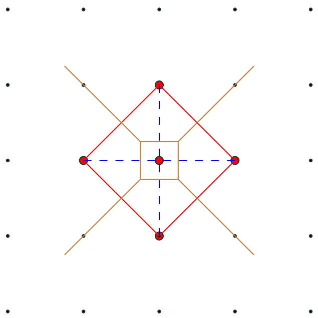
A toric T^3-action is not multi-Hamiltonian for (ω, Ψ). There is, however, subtorus T^2 which preserves Ψ and so is multi-Hamiltonian for the Calabi-Yau structure.

From the T^2-action, we have a moment map,

$$\hat{\mu} : B \to (\mathfrak{t}^2)^* \cong \mathbb{R}^2,$$

which is the projection of μ onto $(\mathfrak{t}^2)^*$.

Toric diagrams and their duals



- 3-dimensional toric, conical Calabi-Yau geometries, $(C(\Sigma), \omega_C, \Psi_C)$, can be described by diagrams in a 2-dimensional lattice, the toric diagram.
- In order to get a corresponding AC picture, (B, ω, Ψ), we choose a triangulation of the corresponding graph.
- The dual graph of the triangulated toric diagram, is the image of the collection of singular T^2 orbits under $\hat{\mu}$.

Toric G_2-manifolds

G_2-structures

M^7 with $\varphi \in \Omega^3(M)$ pointwise linearly equivalent to

$$\varphi_0 = e^{123} - e^1(e^{45} + e^{67}) - e^2(e^{46} + e^{75}) - e^3(e^{47} + e^{56}) \in \Lambda^3(\mathbb{R}^7)^*$$

$e^{ijk} = e^i \wedge e^j \wedge e^k$. The $\mathrm{GL}(7, \mathbb{R})$ stabiliser of φ_0 is $\mathrm{G}_2 \leqslant \mathrm{SO}(7)$.
It determines metric g and orientation vol_g via

$$6g(X, Y)\,\mathrm{vol}_g = (X \lrcorner\, \varphi) \wedge (Y \lrcorner\, \varphi) \wedge \varphi.$$

So we also have 4-form $*\varphi$.

For model form φ_0, $g_0 = (e^1)^2 + \cdots + (e^7)^2$, $\mathrm{vol}_0 = e^{1234567}$ and

$$*\varphi_0 = e^{4567} - e^{23}(e^{45} + e^{67}) - e^{31}(e^{46} + e^{75}) - e^{12}(e^{47} + e^{56}).$$

Holonomy of g is in $\mathrm{G}_2 \iff d\varphi = 0$ and $d *\varphi = 0$.

Toric G_2-manifolds [MS19]

Consider a G_2-manifold (M, φ) with effective T^3 action that is multi-Hamiltonian for both φ and $*\varphi$.

Let U_1, U_2, U_3 generate the torus action. So $\varphi(U_1, U_2, U_3) = 0$ and multi-moment map $(\nu, \mu) = (\nu_1, \nu_2, \nu_3, \mu)\colon M \to \mathbb{R}^4$ satisfies

$$d\nu_1 = \varphi(U_2, U_3, \cdot), \quad d\nu_2 = \varphi(U_3, U_1 \cdot), \quad d\nu_3 = \varphi(U_1, U_2, \cdot)$$
$$d\mu = *\varphi(U_1 \wedge U_2 \wedge U_3, \cdot).$$

At a point p, we can write

$$\varphi = e^{123} - e^{145} - e^{167} - e^{246} - e^{275} - e^{347} - e^{356}$$
$$*\varphi = e^{4567} - e^{23}(e^{45} + e^{67}) - e^{31}(e^{46} + e^{75}) - e^{12}(e^{47} + e^{56}).$$

Moreover, for $p \in M_0$, we can choose our G_2-basis s.t. $\mathrm{Span}\{U_1, U_2, U_3\} = \mathrm{Span}\{E_5, E_6, E_7\}$.

Hence, $(\nu, \mu)\colon M_0 \to \mathbb{R}^4$ has full rank and multi-moment map locally exhibits M_0 as principal T^3-bundle over $\mathcal{U} \subset \mathbb{R}^4$.

Where action is free: Toric G_2

We have that M_0 is the total space of a principal T^3-bundle with connection 1-forms $\theta_1, \theta_2, \theta_3 \in \Omega^1(M_0)$ that satisfy

$$\theta_i(U_j) = \delta_{ij}, \quad \theta_i(X) = 0, \quad \text{for all } X \perp \mathrm{Span}\{U_1, U_2, U_3\}.$$

On M_0 we can define a positive definite symmetric 3×3-matrix of functions by:

$$V = (g(U_i, U_j))^{-1}.$$

Can then write toric G_2-structure in a way resembling Gibbons-Hawking ansatz for gravitational instantons:

$$g = \frac{1}{\det V}\theta^t \operatorname{adj}(V)\theta + d\nu^t \operatorname{adj}(V) d\nu + \det(V) d\mu^2$$

$$\varphi = -\det(V) d\nu_{123} + d\mu \wedge d\nu^t \operatorname{adj}(V)\theta + \underset{ijk}{\mathfrak{S}}\, \theta_{ij} \wedge d\nu_k$$

$$*\varphi = \theta_{123} d\mu + \frac{1}{2\det(V)}\left(d\nu^t \operatorname{adj}(V)\theta\right)^2 + \det(V) d\mu \wedge \underset{ijk}{\mathfrak{S}}\, \theta_i \wedge d\nu_{jk}$$

Note that G_2-structures defined by the above formulae are generally not torsion-free, so holonomy reduction is not guaranteed.

Torsion-free condition amounts to following system of PDEs:
$V \in \Gamma(\mathcal{U}, S^2(\mathbb{R}^3))$ is a positive definite solution to

$$\sum_{i=1}^{3} \frac{\partial V_{ij}}{\partial \nu_i} = 0 \quad \text{for each } j = 1, 2, 3 \qquad \text{(Divergence-free)}$$

and

$$L(V) + Q(dV) = 0 \qquad \text{(Elliptic)}$$

where

$$L = \frac{\partial^2}{\partial \mu^2} + \sum_{i,j=1}^{3} V_{ij} \frac{\partial^2}{\partial \nu_i \partial \nu_j}$$

and

$$Q(dV)_{ij} = -\sum_{a,b=1}^{3} \frac{\partial V_{ia}}{\partial \nu_b} \frac{\partial V_{bj}}{\partial \nu_a}.$$

Can produce solutions for special ansätze but these are generally incomplete.

Global local coordinates

Theorem ([MS19])
For toric G_2-manifolds the multi-moment map induces a local homeomorphism $M/G \to \mathbb{R}^4$.

For complete examples to be constructed below, multi-moment maps provide natural *global* identification of the orbit space with $\mathbb{R}^4$.

Note difference from (compact) symplectic case. Above, orbit space is manifold (without corners).

What is the equivalent of a Delzant polytope in this setting?

Combinatorial data: Image of singular locus

Recall that for toric G_2, we have:

$$d\nu_1 = \varphi(U_2, U_3, \cdot), \quad d\nu_2 = \varphi(U_3, U_1 \cdot), \quad d\nu_3 = \varphi(U_1, U_2, \cdot)$$
$$\text{and} \qquad d\mu = *\varphi(U_1 \wedge U_2 \wedge U_3, \cdot).$$

If, say, U_3 vanishes on a collection of singular orbits, then ν_1, ν_2 and μ are constant on that collection and we get a line segment parameterised by ν_3.

In general, we get the following:

- S^1 stabilisers correspond to lines in $\mathbb{R}^3 \times \{\mu = \mathrm{const}\} \subset \mathbb{R}^4$ of rational slope.
- T^2 stabilisers correspond to points in $\mathbb{R}^3 \times \{\mu = \mathrm{const}\} \subset \mathbb{R}^4$, with 3 incoming vertices.
- "Zero tension" condition on slopes at each vertex.

There are no fixed points and no exceptional orbits.

Producing toric ALC G_2-manifolds

Constructing examples

Theorem ([FHN21])

Let (B, ω, Ω) be an AC Calabi-Yau 3-fold and $M \to B$ a (non-trivial) circle bundle such that $c_1(B) \cup [\omega] = 0 \in H^4(B)$. Then there exists a 1-parameter family, φ_ϵ, of parallel G_2-structures on M.

Each (M, φ_ϵ) constructed in this way has holonomy equal to G_2.

Proposition

If (B, ω, Ω) is toric, then so is each (M, φ_ϵ).

Key is that the multi-toric T^2-action on B lifts to a commuting action on M. Expressing φ_ϵ as a series expansion, one can check that the resulting T^3-action is multi-Hamiltonian for each φ_ϵ.

Corollary

There are infinitely many distinct families of toric G_2-manifolds.

Trivalent graphs

Given an AC Calabi-Yau (B, ω, Ψ) and a circle bundle M over it, the infomation needed to compute

$$c_1(B) \cup [\omega] \tag{1}$$

is encoded in the toric diagram. We have to compute integrals of the form

$$\int_E F \wedge \omega,$$

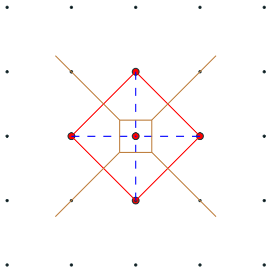
where $F \in \Omega^2(B; \mathbb{Z})$ is a representative of c_1 and E a compact divisor of B. This amounts to computing triple intersections between divisors.

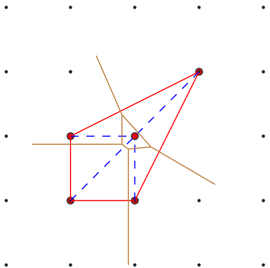
If (1) vanishes in $H^4(B)$, similar computations allow us to construct the trivalent muli-moment graph by "lifting" the planar graph of $\hat{\mu}\colon B \to (\mathfrak{t}^2)^*$ (dual of the toric diagram).
Then, (1) is equivalent to saying that a loop in the planar graph remains a loop when lifted to the trivalent graph.

New complete examples

Quadrilaterals, known examples [FHN21, AFNS21, Fos21]

Up to $\mathrm{GL}(2,\mathbb{Z})$ equivalence, there are exactly 2 relevant toric quadrilateral diagrams with 1 compact divisor of relevance to our construction:

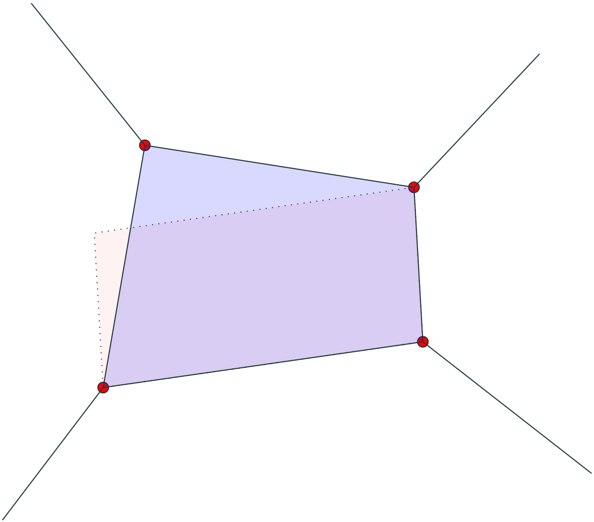




Note the toric diagram on the right admits another triangulation.

Both of the above give rise to infinitely many different toric G_2-manifolds.

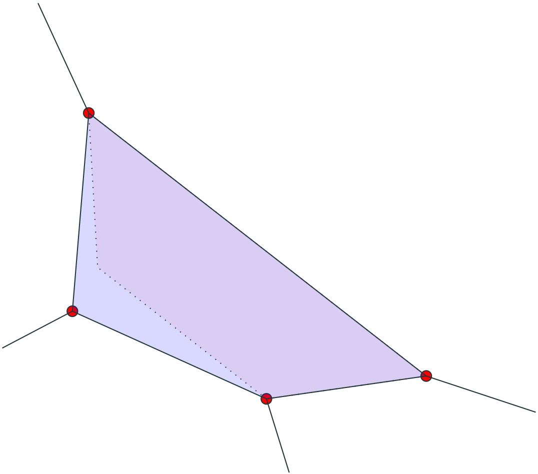
Lifted symmetric quadrilateral: $M_{m,n}$ family



$$c_1 \cup [\omega] = 0 \iff bp + aq = 0$$

$$a, b > 0 \text{ and } p, q \in \mathbb{Z}.$$

Lifted skew quadrilateral: Bundle over resolved $C(Y^{2,1})$

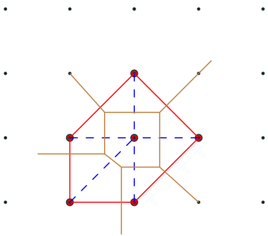


$$c_1 \cup [\omega] = 0 \iff -ap + bp + aq = 0$$

$$a > b > 0 \text{ and } p, q \in \mathbb{Z}.$$

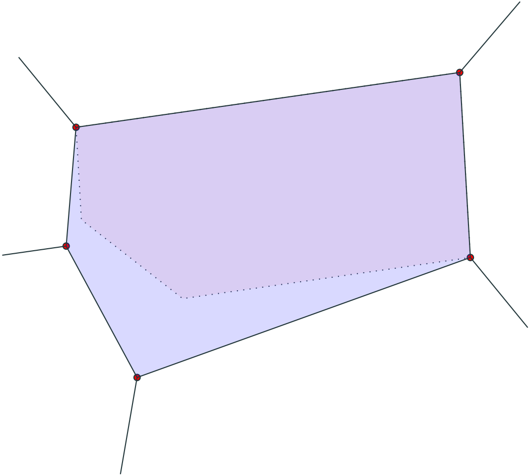
Pentagon

Up to $\mathrm{GL}(2, \mathbb{Z})$ equivalence, there is only 1 relevant toric pentagon diagrams with 1 compact divisor of relevance to our construction:



Note another triangulation is possible.

Lifted graph, pentagon: New family of examples

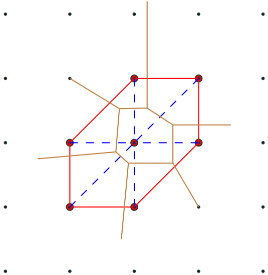


$$c_1 \cup [\omega] = 0 \iff -ap + bp + aq - bq + cq + br = 0$$

$$b > a > 0, c > b - a \text{ and } p, q, r \in \mathbb{Z}.$$

Hexagon: New family of examples

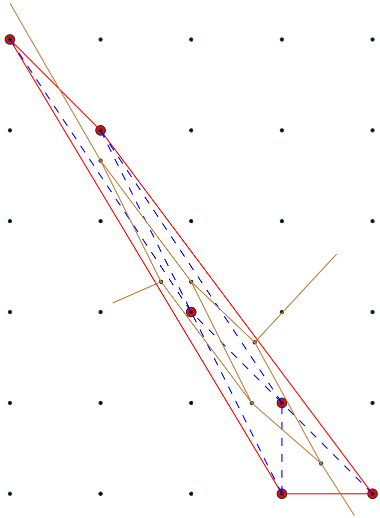
Up to $\mathrm{GL}(2,\mathbb{Z})$ equivalence, there is only 1 relevant toric hexagon diagrams with 1 compact divisor of relevance to our construction:



$$c_1 \cup [\omega] = 0 \iff -ap + bp + aq - bq + cq + br - cr + dr + cs - ds = 0$$

$$a, d > 0, \quad 0 < a < b, \quad c > b - a, \quad d > c.$$

Bundle over resolved cone of $L^{1,5,2}$



$$c_1 \cup [\omega] = 0 \iff bp + aq + 2bq + cq + br = 0 \quad \text{and} \quad ap - cr = 0$$

$$b, -c > 0 \quad \text{and} \quad a > |c|.$$

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